AN ELEMENTARY COURSE

OF

THEORETICAL AND APPLIED MECHANICS.

AN ELEMENTARY COURSE OF THEORETICAL AND APPLIED MECHANICS

DESIGNED FOR

THE USE OF SCHOOLS, COLLEGES,

AND

CANDIDATES FOR UNIVERSITY AND OTHER EXAMINATIONS.

BY

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PREFACE.

This Introduction to the Study of Theoretical and Applied Mechanics contains exact but simple demonstrations of all the propositions usually given in similar elementary treatises, with numerous experimental illustrations and practical applications. It is divided into sections, which are to a great extent independent. Each section is followed by a collection of examples, either original or taken from examination papers.

It is not necessary that all the chapters should be taken in order; students reading for the First Examination for B.Sc. of the London University may omit Chapter XII. in Statics, and sections 8 to 11, 16, 17, 23 to 30, 41 to 44, and 51 to 62 in Dynamics; but these parts are required for the Second Examination for B.A. and B.Sc.

The methods adopted in some parts of the work are different from those of other similar treatises; for instance, the fundamental propositions on the Centre of Gravity are based on the principle of limits; and in

the propositions on Motion the relations between the variable elements have been expressed geometrically, so that the demonstrations in Dynamics, as well as in Statics, are geometrical rather than algebraical.

Although the work is specially adapted to the curriculum of the University of London, the author has endeavoured to make it a useful text-book for schools generally, and for students preparing for other examinations.

SECOND EDITION.

THE Second Edition, besides being a thorough revise of the First, contains additions on the following subjects: Forces not in the same plane; the Mechanical Advantage of Compound Machines; Newton's Third Law of Motion; Energy and the Relation between Force and Energy.

April, 1871.

CONTENTS.

MECHANICS.

I.—INTRODUCTION.

1. A BODY is *at rest* when it constantly keeps the same position in space, and it is *in motion* when it occupies successively different positions.

2. To illustrate these conditions, let us place a stone or other body on a horizontal table. If left to itself, the stone will remain for an indefinite length of time in the same position.

Imagine the table to be suddenly removed; the stone will no longer remain at rest, but will fall to the earth. Why does it take that particular direction? A physicist would tell us that there is a constant attraction between the earth and the stone, and that the movement of the stone is due to this cause.

Whenever a body at rest is made to move, there always exists a cause of the motion, the action of which is due to the presence, near or remote, of another body; in other words, a body without life cannot move of itself.

3. If the stone be projected along a level road, the speed with which it leaves the hand will not be maintained, but will be gradually diminished, until finally

the stone will stop in its course. If, instead of the road, the frozen surface of a lake be chosen, the same stone thrown with the same force will travel much further on the ice than on the road. And if, instead of the irregular stone, we roll a smooth ball of ivory on the ice, the distance traversed will be greater still. It is evident, therefore, that the stone is gradually stopped by the resistances it encounters.

Similarly, whenever a body ceases to move, it does so because its motion is destroyed by the resistances it meets with. The more we diminish these resistances, the longer and the further will the body move; and, consequently, if we imagine that they are all suppressed, we shall be led to the conclusion that the body under these circumstances would continue to move for an indefinite length of time; in other words, that a body cannot of itself alter its speed.

4. Neither can it change the direction of its motion. If no obstacle be encountered in its course, the ivory ball thrown on the ice will turn neither to the right nor to the left. It is true that a stone thrown into the air returns to the ground, but this is because its weight tends constantly to bring it to the earth. Conceive the weight and the resistance of the air removed, and the stone will continue to move in a straight line with uniform speed. This fundamental principle in Mechanics is termed the First Law of Motion, and may be enunciated thus:—

When a body is not acted on by any external agent, if it be in a state of rest it will remain so, and if it be in motion it will continue to move in a straight line with uniform speed.

Force.

5. When we raise a burden, set in motion a body at rest, or arrest a body already in motion, we are conscious of exerting a certain effort, and the term force, which we give this effort, is naturally applied to similar causes.

A force is therefore any cause tending to move a body to change its motion, or to keep it at rest when other forces are acting upon it.

For a long time the forces employed by man were only those furnished by human and animal labour; but, in proportion as progress in science has been made, we have not only utilised these natural mechanical forces, but have subjected to our use others which were before unknown; as, for example, forces arising from the motion of the air, a running stream, the changing of water into vapour by heat, chemical action, electricity, &c.

6. Forces from different sources, however, may be compared with one another, and are capable of numerical valuation.

The science of Mechanics does not consider the nature of the forces, but treats only of the properties common to all; and a force will be regarded as fully determined when we know the point at which it is applied, its direction, and its intensity.

7. Forces do not always have the effect of producing or modifying the motion of a body; other forces acting at the same time may counteract them. A weight held in the hand is not the less heavy because it does not fall; it is prevented from falling by a force exactly

equal to it, but opposite in direction. If the weight be placed on the table, it is still prevented from falling by a force exerted by the table. This force is termed a *reaction*, or a *resistance.* Whenever a body acted on by a force is at rest, we may at once conclude that there exists a resistance which counteracts the force. When two or more forces neutralise one another, so as to keep the body or the particle on which they act at rest, they are said to be *in equilibrium.*

8. The science of Mechanics may be conveniently divided into two parts.

Statics treats of forces in Equilibrium.

Dynamics treats of forces which produce motion.

II.—STATICS.

9. Three things have to be considered in a force.

1st. Its intensity or magnitude.

2nd. The point of application.

3rd. The direction.

The Magnitude of a Force.

10. Two forces are equal when, if applied at the same point in opposite directions, they will be in equilibrium. One force is twice another when the first will counteract two forces, each equal to the second, applied to the same point, but in the opposite direction; so also one force is thrice another when three of the latter

will be required to keep the former in equilibrium, and so on.

11. The magnitude of a force is estimated by stating how many times a given unit will be equal to it. The unit usually employed is a pound. Although forces are obtained from different sources, they may all be compared with the pound weight.

We will describe an instrument which may be used to compare forces of different kinds with the unit of weight.

12. Figure 1 represents a spring balance for this purpose, consisting of an elastic band of steel, B O A, to the ends of which metallic graduated arcs are attached. The outer arc, D A, is fixed to the lower arm, passes through an aperture in the upper, and terminates in a ring, F. The inner arc is attached to the upper arm, passes freely through the lower, and terminates in a hook, G. Place the ring on a firm support, and hang a weight of one pound upon the hook. The two branches of the steel band will approach each other until the elasticity of the bar counterbalances the weight. Mark on the arc, A D, the position of the bar. Suspend in a similar manner weights of 1 lb., 2 lbs., 3 lbs., &c., and make the corresponding marks until the

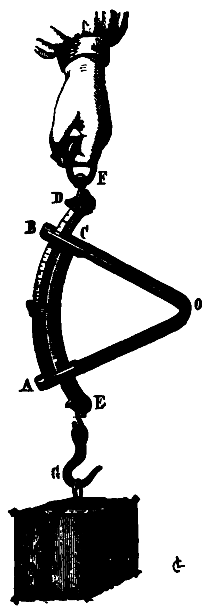


Fig. 1.

graduation of the arc is completed. Any force which when applied to the instrument produces a certain deflection is equal to the weight which would produce the same.

A spring balance used to compare forces differing in kind may be termed a *dynameter.*

Figures 2 and 3 represent other forms of the instrument. If the ends of the bar, A O B in Fig. 1 were


Fig. 2.

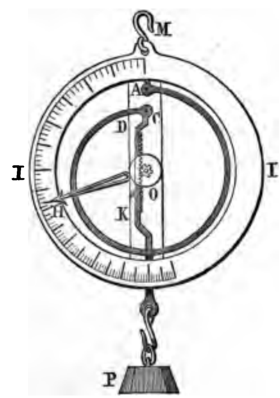


Fig. 3.

united to a similar bar, by joints at A and B, the strength of the instrument would be greatly increased, and by means of such an arrangement we should be able to compare even forces of great magnitude with the unit of weight. If we wish, for example, to measure the force with which a horse draws a carriage along a paved road, it will only be necessary to interpose the dynameter between the horse and carriage, attaching one to the ring, and the other to the hook.

The Direction of a Force.

13. When a body is suspended by a thread (Fig. 4), the thread takes a determined direction, termed the vertical; if it be suddenly cut, the body will fall in the direction of the vertical. This is therefore termed the direction of the weight.

The direction of a force applied to a point is the straight line along which the point would be displaced if it were free to move.

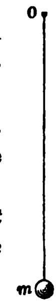


Fig. 4.

Graphic Representation of Forces.

14. In the three properties we have described, forces may be represented by straight lines. For example, the direction of the line may represent the direction of the force, the extremity of the line the point of application of the force, and if we agree to represent a unit of weight by a unit of length, as for instance, by taking a line of three inches to represent a force of three lbs., then the magnitude of the line will represent the magnitude of the force.

The Transmissibility of Forces.

15. If a weight be attached by a cord to the spring balance in Fig. 1, the effect will be the same at whatever point in the cord the weight be tied. Similarly, a force may be applied to a body directly, or by the interposition of a rigid rod, and, supposing the rod to be supported independently, the result will be the

same. The general principle here illustrated may be stated thus. *A force may be applied at any point in the line of its direction, provided this point be connected with the first point of application by a rigid and inextensible straight line.*

Let A B (Fig. 5) be a string, at the extremities of which equal forces are acting in opposite directions, it

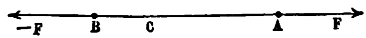


Fig. 5.

is clear that the string will be in equilibrium. Take any other point, C, in the string, and remove the force, F, from A to C, still the force will be in equilibrium. Hence we may consider the force applied at one end to be transmitted throughout the string, and we may suppose two opposite forces at any point, each equal to F. Either of these forces is termed the *tension* of the string. Suppose the string to pass round a smooth peg, ring or surface, in this case also *the tension of the string is the same at every point.*

Resultant of Forces in the same Straight Line.

16. If we hang two weights of 1 lb. each to the dynameter, the indication of the instrument will be precisely the same as if a single 2 lb. weight were attached to it. Three weights of 1 lb. each produce the same effect as a single 3 lb. weight. Similarly, forces of 3 lbs. and 5 lbs. may be replaced by a single force of 8 lbs. And, generally, it is evident that *when two or more forces act upon a point in the same straight line*

in the same direction, their effect will be equivalent to that of a single force equal to their sum.

Conversely, if for a single force two others, the sum of which is equal to the first, be substituted, the effect will be the same.

Let us now apply at the point O two unequal forces in opposite directions, the one, F′, acting horizontally from left to right, and the other, F, horizontally from right to left (Fig. 6). If F′ be a force of 2 lbs. and F of 3 lbs., we may substitute three single forces of

Fig. 6.

1 lb. each for the latter, and two of them will then be in equilibrium with the former. There remains a free force of 1 lb. acting from right to left. The body under the influence of the two forces is therefore in the same condition as if it were acted on by a force, F—F′, in the direction of the greater. Whenever two unequal forces act on a particle in the same straight line, but in opposite directions, the greater may be divided into two parts, one of which is equal to the less, and will therefore neutralise it, and an effective force will remain in the direction of the greater, equal to the difference of the forces.

The single force which will produce the same effect as several forces acting together, is termed the resultant of the forces; hence the general proposition explained above may be enunciated thus.

When any number of forces act upon the same point, in the same straight line, their resultant may be found

by taking the difference between the sum of all the forces acting in one direction, and the sum of all the forces acting in the other, the direction of the resultant being that of the greater sum.

Let us agree to write a + sign before all the forces acting in one direction, and a — sign before all those acting in the opposite direction, and then we may state the rule thus:—*The resultant of a number of forces acting in the same straight line is equal to the algebraical sum of the forces.*

17. The forces which are thus combined are termed the *Components*, and the process of finding the resultant is termed the *Composition of Forces.* The converse operation, namely, that of finding Component forces which shall have a given *resultant*, is termed the *Resolution of Forces.*

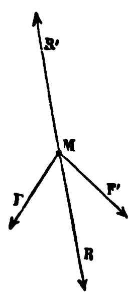


Fig. 7.

The resultant is the single force which will produce the same effect as the components; hence, if to a system of forces a new force be added, exactly equal to the resultant, and opposite in direction, this force will keep the system in equilibrium (Fig. 7). Thus the problem to find the resultant also gives the force required to keep the system in equilibrium.

Exercises.

1. If a force which will support a weight of 18 lbs. be represented by a line 3 ft. long, how long must be the line which will represent a weight of $1\frac{1}{2}$ cwts. ?—*Ans.* 28 ft.

2. In the above case, what weight will a line of 17 in. represent ?—*Ans.* $8\frac{1}{2}$ lbs.

3. The force required to balance two others acting together in the same straight line is five times one-fourth of larger force, what multiple is it of the smaller ?—*Ans.* 5.

4. When a ton is represented by a line 5 ft. 4 in. long, what force will a line 7 in. represent ?—*Ans.* 2 cwt. 21 lbs.

5. The weight of a cubic foot of a substance is 6 cwt., and is represented by a line a foot long, what will be the length of the line required to represent 720 cubic inches of the same substance ? —*Ans.* 5 inches.

6. When two forces act together they have a resultant of 12 lbs., and when they act in opposite directions their resultant is 2 lbs.; find the forces.—*Ans.* 5 lbs. and 7 lbs.

7. A string supports a weight of 4 lbs. at its extremity, another weight of 5 lbs. above the first, and a third of 7 lbs. above the second: find the tensions of the three parts of the string. If the tension of the middle part be represented by a line $11\frac{1}{4}$ in. long, what will be the length of the lines required to represent the others ?—*Ans.* Tensions, 4 lbs., 9 lbs., 16 lbs.; lines, 5 in. and 20 in.

III.—THE COMPOSITION AND RESOLUTION OF FORCES.

Forces in the same Plane meeting at a Point.

18. Suppose two forces not in the same straight line act on a point M (Fig. 8). If left to the action of these forces only, the point will begin to move in a certain direction. There must, therefore, be a single force which will produce the same effect, and therefore, also, a force which will counteract this effect. We have to consider how we can find the magnitude and direction of this force.

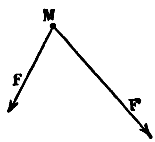


Fig. 8.

Over two pulleys, M and N (Fig. 9), capable of moving without friction, pass a fine thread of silk. To the middle point B tie a very light strip of wood E D. Take a number of small equal weights, and attach two of them to each end of the thread, and three to the extremity E of the strip E D. The point B will descend to a definite position, and will then remain at rest.

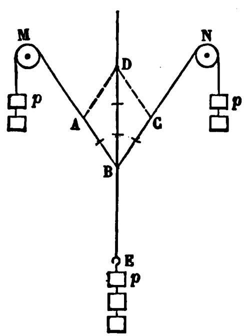


Fig. 9.

Let us examine the position of equilibrium. First, we shall find that the rod E D will bisect the angle M B N; and this will always be the case whatever the

weights, provided that those at M and N are equal. Again, take a certain length—an inch, for example—to represent the weight p, and measure off on B N and B M as many inches as there are weights at each end of the thread and through the points A and C thus determined, draw A D and C D parallel to B C and A B respectively. These lines will meet the rod in the same point D, and it will be found that the line B D will contain as many inches as there are weights attached to E.

Instead of equal weights suspend 3 p at one end of the thread, 2 p at the other, and 4 p to the rod (Fig. 10). The form of the figure will be changed, but if we measure on B M three units of length, and draw A D parallel to B C, also measure on B C two units, and draw C D parallel to B A, still we shall find that these straight lines intersect at a point D on the rod, and that the line B D contains as many units of length as there are weights at E.

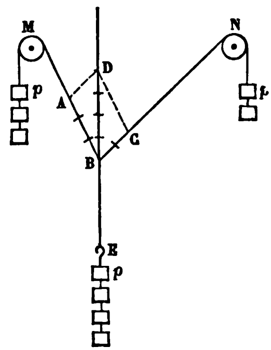


Fig. 10.

This experiment illustrates the fact, that, in order to find the resultant of two forces acting on a point, we must take on their directions lengths proportional to the forces, and complete on them a parallelogram. The diagonal of the parallelogram through the point of application will represent the resultant of the forces in magnitude and direction.

19. We may here remark that the thread must be

tied to the rod, in order that there may be equilibrium with unequal weights at M and N. If the weight at B be attached by a ring, as in Fig. 11, the tension in

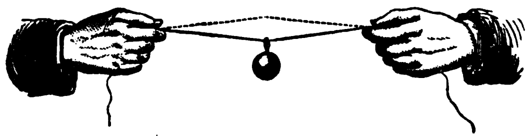


Fig. 11.

the two parts of the string must be equal. The parallelogram in this case is equilateral.

The proposition above explained is termed the Parallelogram of Forces.

20. *If two forces acting on a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant of these forces will be represented in magnitude and direction by that diagonal of the parallelogram which passes through the particle.*

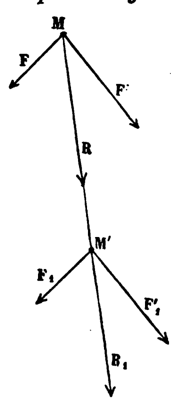


Fig. 12.

Before proceeding to the proof of this proposition it is necessary to notice a preliminary proposition.

21. Any two forces, F, F′, applied at a point M may be transferred parallel to themselves to any other point M′ in the line of direction of the resultant (Fig. 12). We may substitute for the forces their resultant, and transfer it to the point M′ in the line of its direction, and may then resolve it at that point into forces F and F′ equal and parallel to the original components.

To demonstrate the Parallelogram of Forces for the Direction of the Resultant.

22. *1st. When the forces are equal.*

It is evident that the resultant of two equal forces F and F′ acting on a point M bisects the angle between them; for there is no reason why it should lie nearer one than the other (Fig. 13). This reason may be

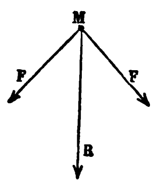


Fig. 13.

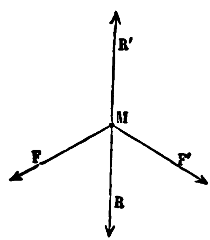


Fig. 14.

termed a *conclusion from symmetry.* A similar reason would lead to the conclusion that three equal forces acting on a particle, and making with each other angles of 120°, are in equilibrium (Fig. 14).

2nd. When one force is a multiple of the other.

Let M A and M B represent two forces applied at the point M, and suppose M A to contain M B a certain number of times, three for instance (Fig. 15). Construct the parallelogram M A M′B. Decompose the force represented by M A into three forces, each equal to M B, represented by the lines M C, C D, D A. Since M B and M C are equal, E is a point in their resultant, and we may transfer the forces parallel to themselves to the point E, so that one will act in the line E G, and the other in

C B′. Consider the latter and the force represented by O D. Since they are equal, G is a point in their resul-

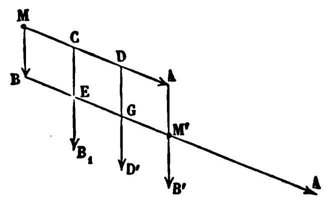


Fig. 15.

tant, and we may transfer the forces to the lines, B M′ and D D′. Combine the forces represented by D D′, and D A, and transfer to M′ as before. We have now the three components of M A transferred to B M′, and M B removed to M′B′. The directions of all these pass through the point M′, and consequently their resultant passes through the point M′; but it also passes through the point M, and consequently it has the direction of the diagonal M M′.

3rd. When the forces have a common measure.

Let the forces represented by M A and M B have a common measure *f* (Fig. 16). Divide M B into parts equal to this common measure. Let M C be the first of these parts. By the last proposition the resultant of M A and M C passes through D, so that we may transfer M A to C D parallel to itself, and M C to A D parallel to itself. Combine the force represented by C D with C B, the next portion of M B. As before, M′ is a point in the resultant, and we may transfer the forces to this point. If there were more divisions of M B, we should proceed in the same way with all. The original forces are thus

replaced by others passing through the point M′. This point is therefore in the direction of their resultant,

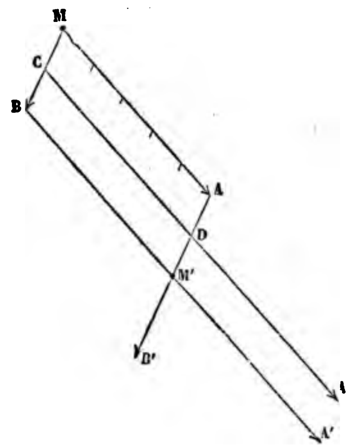


Fig. 16.

hence the diagonal M M′ of the parallelogram has the direction of the resultant.

4th. When the forces are incommensurable.

If the forces have no common measure, choose a small force, f, which is contained exactly in one of the forces F, and let P equal to $n \times f$ be a force less than F′, and Q equal to $(n+1) \times f$ be greater than F′. Then the proposition has been proved for F and P, also for F and Q. Now by taking f sufficiently small, we may make P and Q differ from each other, and therefore differ from F′, which lies between them by a quantity as small as we please; hence we conclude, generally, that if two forces acting at a point be represented in magni-

tude and direction by two adjacent sides of a parallelogram, their resultant will have the direction of the diagonal of the parallelogram through that point.

To prove that the Parallelogram of Force is true also with respect to the magnitude of the Resultant.

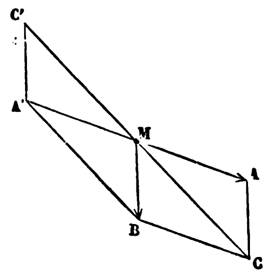


Fig. 17.

23. Let M A, M B, be the direction of the forces (Fig. 17). Apply at the point M a force represented by M C′ which will be in equilibrium with the given forces. Then any one of the three forces, M A, M B, M C′ will be equal and directly opposite to the resultant of the other two, consequently by the last proposition each must be in the same straight line as the diagonal of the parallelogram on the other two. Complete the parallelograms on M A and M B, and on M C′ and M B, therefore C C′ and A A′ are straight lines, and A′B C M is a parallelogram. Therefore C′M and C M are each equal to A′B and to one another, hence C M has the magnitude as well as the direction of the resultant of B M and A M.

The parallelogram of forces is therefore completely proved.

24. In working the following Exercises the student will be frequently required to solve the following triangles :—

1. A right-angled triangle in which the acute angles are

respectively 30° and 60°. Such a triangle is half an equilateral triangle, and it is easily proved that

The shortest side = half the longest.

The third side = $\sqrt{3}$ × the shortest.

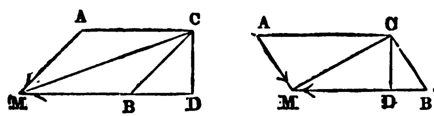
2. An isosceles right-angled triangle (in which of course the angles are 45°, 45° and 90°).

The hypothenuse = either side × $\sqrt{2}$.

To find the diagonal of the parallelogram when the angle is 60°, 30°, or 45°.

If C M be the diagonal (Fig. 18*a*), produce a side M B to D, and draw C D perpendicular to M D. The angle C B D = A M B. If this angle be 60°, 30°, or 45°, and M B and C B be given, C D and B D may be found as above, and $CM = \sqrt{(MD^2 + CD^2)}$ $= \sqrt{(MB^2 + BC^2 + 2\,MB \cdot BD)}$.

To find the diagonal when the angle is 120°, 150°, or 135° (Fig. 18*b*).



Figs. 18*a* and 18*b*.

In this case the angle B is 60°, 30°, or 45°, and therefore if C D be drawn perpendicular to M B, C D, D B, and M D may be found, and $CM = \sqrt{(MD^2 + CD^2)} = \sqrt{(MB^2 + BC^2 - 2\,MB \cdot BD)}$.

Remark that the unit of length taken to represent a unit of force may be anything we please, but when a certain line has been taken to represent a given force, the unit is fixed, and must be maintained throughout the problem.

Example.—A boat is held at rest in a stream by cords, A O, B O (Fig. 18), attached to stakes in the opposite banks, and a rope, C O, fastened to the boat C. Show how to find the ratio of the tensions in the cords.

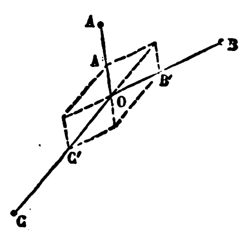


Fig. 19.

Take *any* length O A′ to represent

the tension in O A. Through A′ draw a line parallel to O B, and produce C O to meet it. From the point of intersection draw a parallel to O A, meeting O B in B′. The ratio of O A′ to O B′ will be the ratio of the tensions. The parallelogram might have been constructed on any two of the lines. If the absolute value of one of the tensions be known, as for example that in A O = 50 lbs., the unit taken $\left(= \frac{A'O}{50}\right)$ is known, and the tension in the other cords may be found.

Exercises.

1. Three ropes, P A, Q A, R A, are knotted together at the point A; P A is attached to a tree, Q A and R A are pulled by two men; having given the angle between Q A and R A and the force exerted by each man, show how to find the pressure on the tree.

2. At what angle must two equal forces act to produce the same effect as one of them?

3. Two rafters, making an angle of 60°, support a chandelier weighing 90 lbs; what will be the pressure along each rafter? —*Ans.* $30\sqrt{3}$ lbs.

4. For a given vertical force two other forces are to be substituted, one horizontal and the other making an angle of 45° with the vertical; find the magnitude of the forces.

5. If a body be drawn along the ground by a cord inclined at a given angle, show how to find what part of the force is spent in lifting the body, and what part in drawing it.

6. A crane is used to raise a weight, as represented in the figure; a man pulls the weight by a cord to keep it from the wall.

(*a*) Show that the tension in the cord above the weight is greater than the weight, and show how to find the ratio between them.

(*b*) Show how to determine the direction and magnitude of the pressure on the pulley of the crane.

7. Forces of 8 cwt. and 15 cwt. act on a point at right angles; find their resultant.—*Ans.* 17 cwt.

8. The ratio of two forces acting at right angles is $\frac{3}{4}$, and the smaller is 21 lbs.; find the resultant.—*Ans.* 35 lbs.

9. The resultant of two forces which act at right angles is 145 lbs. and one of the forces is 144 lbs.; find the other.—*Ans.* 17 lbs.

10. The resultant of two forces which act along the adjacent sides of a square is 169 lbs., and one force is 1 lb. greater than the other; find the forces.—*Ans.* 119 and 120.

11. Show that the greater the angle between two forces, the less will be their resultant.

12. Two forces, P and Q, act upon a point; $P=3Q$. If two-thirds of P were taken and 8 lbs. were subtracted from Q, the resultant would have the same direction as before. Find the forces.—*Ans.* $P=72$; $Q=24$ lbs.

13. Two equal forces of $2\sqrt{3}$ lbs. act at an angle of 60°; find their resultant.—*Ans.* 6 lbs.

14. Two equal forces act at an angle of 120°; prove that the resultant equals either of the forces.

15. The resultant of two forces at right angles is $8\sqrt{3}$ lbs. and makes an angle of 30° with the larger force; find the forces. —*Ans.* $4\sqrt{3}$, and 12 lbs.

16. Forces of 8 and 12 lbs. act at an angle of 60°; find the resultant.—*Ans.* 17·43.

17.—Forces of 9 and 11 lbs. act at an angle of 120°; find their resultant.—*Ans.* $\sqrt{103}$ or 10·14.

18. The resultant of two forces, P and Q, is perpendicular to P; show that it is less than Q.

19. Two forces, each equal to 50 lbs., act at an angle of 150°; find their resultant.—*Ans.* 25·8819.

20. Two forces, each equal to 10 lbs., act at an angle of 30°; find their resultant.—*Ans.* 19·319.

21. If the resultant of forces of 33 lbs. and 56 lbs. is 65 lbs., show that the angle between the forces is a right angle.

22. Two forces of 2 and 3 cwts. respectively act at an angle of 45°; find their resultant.—*Ans.* 4·63 cwts.

23. The resultant of two forces, P and Q, acting at right angles is 305, and $\frac{P}{Q} = \frac{11}{60}$; find the forces.—*Ans.* 55 and 300 lbs.

24. The resultant of two forces is perpendicular to one of them; the forces are 2 cwts. 17 lbs. and 1 cwt. 8 lbs.; find the magnitude of the resultant.—*Ans.* 209 lbs.

25. Resolve $17\sqrt{2}$ lbs. into two equal forces at right angles. —*Ans.* 17 lbs.

26. Resolve a force of 353 lbs. into two which shall contain an angle of 60°, and the larger of which shall be $150\sqrt{3}$ lbs.—*Ans.* The smaller = 142·0962 lbs.

27. A force of 205 lbs. is resolved into two, which make an angle of 30°. The larger force is 168 lbs.; find the smaller.—*Ans.* 41·507732 lbs.

28. Can there be equilibrium with forces of 4, 5, and 9 lbs. acting on the same point?

29. A and B are points in a horizontal line to which is attached a thread bearing at a point C a weight of 194 lbs. If A B = 97, A C = 72, C B = 65 in.; find the tensions in the two parts of the threads.—*Ans.* 130 lbs. and 144 lbs.

IV.—DEDUCTIONS FROM THE PARALLELOGRAM OF FORCES.

Several important deductions may be made immediately from the Parallelogram of Forces.

The Triangle of Forces.

25. *When three forces acting on a particle can be represented in magnitude and direction by the three sides of a triangle taken in order, they will be in equilibrium.*

Let M A C (Fig. 20) be the triangle; complete the parallelogram M A C B, then the force represented by A C is also represented by M B, and these forces have a resultant represented by M C. This resultant will be in equilibrium with the equal and opposite force C M, and therefore the original forces are in equilibrium.

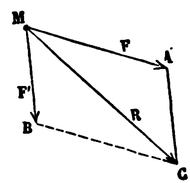


Fig. 20.

Remark in the enunciation the words *taken in order.* The forces act from M towards A, A towards C, C towards M. If one be reversed, they no longer represent forces in equilibrium.

It should be noticed also that the forces are supposed to pass through a point, and the sides of the triangle to be *parallel* to them, so that parallel lines are considered to have the same direction.

26. The converse of this proposition is also true. *When three forces acting on a particle are in equilibrium,*

the sides of any triangle which are parallel to the lines of action of the forces are also proportional to the forces.

Thus, if P, Q and R be forces in equilibrium, represented by M C′, M A, M B, they will be proportional to the sides of the triangle, A M C, and of any triangle formed by lines parallel to these sides.

Again, it is a theorem in geometry that if there be two triangles, such that the sides of one are respectively perpendicular to those of the other, then these sides are proportional, hence in the above proposition, if the lines be drawn perpendicular to the direction of the forces they will be proportional to the forces.

27. From the triangle of forces, it follows that when three forces acting on a point are in equilibrium, the sum of any two is not less than the third. The sum of two may be equal to the third when the latter is opposed to the former, and acts in the same straight line.

28. If we increase the angle between the forces F and F′ (Fig. 20), we *decrease* the angle A, and the new triangle having two sides respectively equal to the two sides, M A, C A, but including a smaller angle, will have a smaller base. Consequently, *the greater the angle between the forces the less will be the resultant*, and it is the least possible, when the forces are in the same straight line, but opposite in direction.

29. If three forces in the same plane, not parallel, are in equilibrium, they pass through the same point.

For if two meet in a point they may be replaced by their resultant, and in order that this resultant may be in equilibrium with the third force, they must be in

the same straight line; hence the line of action of the third force must pass through the intersection of the first two.

30. The sum of the projections of two forces on the line of action of their resultant is equal to the resultant, and the algebraical sum of the projections of the forces on any line whatever is equal to the projection of the resultant on the same line.*

For the first part of the proposition refer to Fig. 20. If a perpendicular be drawn from A to M C, it divides M C into two parts, which are respectively the projections of M A and A C, the latter of which is equal to the projection of M B.

Again, let the lines M B, M C, M A, be projected on any other line C′ K (Fig. 21) C′, B′, A′, and K, being respectively the projections of the points C, B, A, and M.

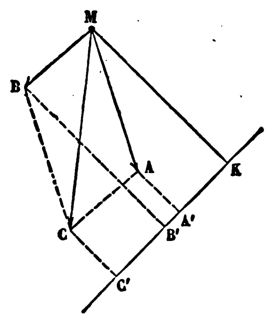


Fig. 21.

First, let the line M K fall without the angle M (Fig. 21). It is evident that

$$C'K = C'B' + B'K$$
$$= A'K + B'K$$

that is, the projection of the resultant is equal to the sum of the projections of the forces.

* The projection of one line upon another is the line included between perpendiculars from the extremities of the first upon the second.

Second, let the line MK fall within the angle M (Fig. 22). Here $C'K = B'K - B'C'$

$$= B'K - A'K.$$

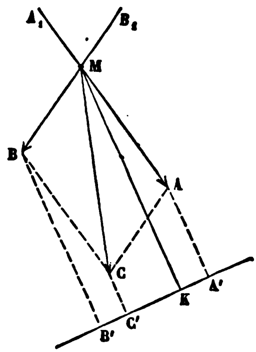


Fig. 22.

If we consider the sign of KA′ to be opposite to the sign of C′K and B′K, because the line is taken in the opposite direction, then B′K — A′K will be the algebraical sum of the projections of the forces. The proposition may be extended to any number of forces. If the forces be in equilibrium the sum of their projections must therefore be zero.

31. If any two lines be drawn at right angles in the plane of the forces, the projections of a given line upon them will represent two forces which have for resultant the force represented by the given line; for the projections are sides of a rectangle of which the given line is the diagonal.

32. From this it follows, that if any number of forces act on a point and two lines be drawn at right angles through the point, the algebraical sums of the projections of the forces on these lines will be equal to the projections of the resultant on the same lines. If we can find these projections we can therefore find the resultant. Let X = the sum of the projections on one line, and Y = the sum of the projections on the other, then $R^2 = X^2 + Y^2$.

The condition that the forces may be in equilibrium is therefore that $X = 0$, and $Y = 0$.

These conditions are sometimes stated thus:—

If any number of forces act on a point, and two lines be drawn at right angles through the point, the sum of the resolved parts of the forces in either direction will be equal to the resolved part of the resultant in the same direction. If the sum of the resolved parts in the two directions be zero, the forces will be in equilibrium.

The Polygon of Forces. To find the resultant of any number of forces acting on a particle.

33. Let F, F', F'', F''' be forces acting on the point M, represented by MA, MB, MC, MD (Fig. 23).

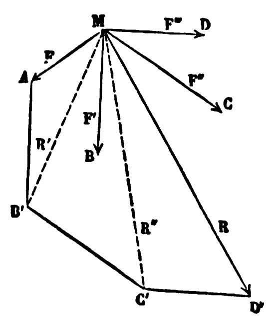


Fig. 23.

Through A draw AB' parallel and equal to MB, then by the triangle of forces MB' will represent the resultant of F

and F′. Through B′ draw B′ C′ parallel to M C, then M C′ will be the resultant of R′ and F″. Similarly, by drawing C′ D′ equal and parallel to M D, we obtain M D′ representing R the resultant of F, F′, F″, F‴. D′M will represent a force which will keep the original forces in equilibrium, and from the construction we may at once deduce the following proposition. When a number of forces acting on a particle can be represented in magnitude and direction by the sides of a polygon taken in order the forces are in equilibrium.

Parallelopiped of Forces.

34. If three forces, F F′ F″, acting on a point be represented in magnitude and direction by the three sides of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal of the parallelopiped through the point of application. The resultant of F and F′ is represented by the diagonal of the parallelogram M B′ (Fig. 24), and the

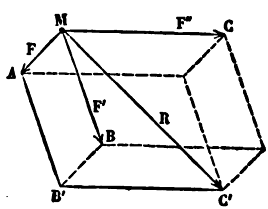


Fig. 24.

resultant of this force and F″ is represented by M C′, the diagonal of the parallelogram M B′ C′ C.

35. The inverse proposition to the composition of forces, namely, the substitution of two or more forces which shall be equivalent to a given force, is termed the *Resolution of Forces.* If the line representing a given force be made the diagonal of a parallelogram the sides of this parallelogram represent forces into which it may be resolved; and since as many parallelograms can be drawn on a given diagonal as we please, a force may be resolved into two others in as many ways as we please.

36. The method of section 31 may be useful when there are several forces acting on a point at known angles. For example:—

Find the resultant of four forces of 2, 3, 4, and 5 lbs., respectively, the included angles being in order 60°, 60°, 15°.

Take two lines, one in the direction of the first force, and the other at right angles to it.

The projections of the first force are ...	2	and	0
" " second " ...	$1\frac{1}{2}$	and	$1\frac{1}{2}\sqrt{3}$
" " third " ...	-2	and	$2\sqrt{3}$
" " fourth " ..	$\dfrac{5}{\sqrt{2}}$	and	$\dfrac{5}{\sqrt{2}}$
" " resultant	$\dfrac{3}{2}-\dfrac{5}{\sqrt{2}}$	and	$\dfrac{7\sqrt{3}}{2}+\dfrac{5}{\sqrt{2}}$

$$R^2 = \left(\frac{3}{2}-\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{7\sqrt{3}}{2}+\frac{5}{\sqrt{2}}\right)^2 = 96\ 2594$$

therefore $R = 9{\cdot}86$

Exercises.

1. If two lines, A B, C A, represent two forces acting on a point, the one towards the point and the other from it, show how to find the resultant.

2. Three forces of 119, 120 and 169 lbs. act on a point and keep it at rest, show that the angle between the first and second is a right angle.

3. Three pegs, A, B and C, driven in a wall form a right-angled isosceles triangle, of which the base, A C, is horizontal. A cord passes over the three pegs and supports two weights of 20 lbs. attached to its ends; find the pressure on the pegs.—*Ans.* Pressure on B $= 20\sqrt{2}$; on A and C $= 20\sqrt{2-\sqrt{2}}$.

4. Three forces acting on a point are represented by lines of 12, 24, and 15 inches, including angles of 60°: find the length of the line which will represent the resultant. (§31)—*Ans.* 37·5899.

5. The angles between three forces of 42, 52, and 10 lbs. respectively, are 120°; find the resultant.—*Ans.* 38 lbs.

6. Show that if three forces acting on a point be represented in magnitude and direction by the three lines drawn from the middle points of the sides of a triangle, to the opposite angles the forces are in equilibrium. (Resolve each force into two others acting along the sides.)

7. Along the sides of an equilateral triangle, A B C, three forces of 1 lb. each act in direction as follows: from A to B, from A to C, and from B to C; find their resultant.—*Ans.* 2 lbs. parallel to A C through the middle point of B C.

8. In the above case, if the forces are 3 lbs. along A C, 5 lbs. along B C, and 7 lbs. along A B, find the resultant.—*Ans.* $\sqrt{84}$.

9. Replace two forces of 20·3 and 39·6 kilogrammes respectively, acting at right angles by two others also acting at right angles, the larger being 40 kilogrammes.—*Ans.* 19·5.

10. Three rods meet at a point and form a tripod to sustain a weight, show how to find the ratios of the pressures on the rods.

11. Three forces, 25, 60, and 72 grammes, having directions at

right angles to one another act upon a point; find their resultant. —*Ans.* 97.

12. A cord is attached to two fixed points, A and B, in the same horizontal line, and bears a ring weighing 10 lbs. at C, so that A C B is a right angle; find the tension in the cord.—*Ans.* $5\sqrt{2}$ lbs.

13. Two cords, A C = 44 inches, B C = 117 inches, are attached to points A and B in the same horizontal line and to a weight of 10 lbs. at C. The angle A C B is a right angle; find the pressure on A and B.—*Ans.* 3·52 and 9·36 lbs.

14. Prove that if two forces be represented by two diagonals of a parallelogram, their resultant will be represented by a line equal to twice one of the sides of the parallelogram.

15. Find the resultant of six forces, 1, $\sqrt{2}$, 2, $3\sqrt{3}$ and 2 lbs. acting on a point, the angles taken in order being 45°, 75°, 60°, 90°, and 90°.—*Ans.* 1 lb. at right angles to the first force.

16. Find the resultant of five equal forces acting on a point and making with each other angles of 60°.

17. Four equal forces act on a point; the first is perpendicular to the second, their resultant is perpendicular to the third, and the resultant of the first three is perpendicular to the fourth; find their resultant.—*Ans.* 2 P.

18. A weight is supported by two strings which are attached to it, and to two points in a horizontal line: if the strings are of unequal length, show that the tension of the shorter string is greater than that of the other.

19. Three forces, represented by those diagonals of three adjacent faces of a cube which meet, act at a point; show that the resultant is equal to twice the diagonal of the cube.

V.—PARALLEL FORCES.

37. Take a bar of iron B C (Fig. 25), and attach it by the middle point D to a fine thread passed over a

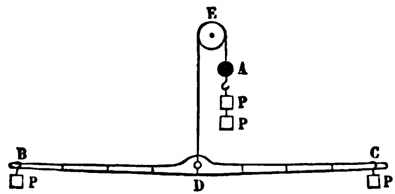


Fig. 25.

pulley E capable of moving without friction. To the other end of the cord attach a mass of lead and a hook A, which will exactly balance the bar. Take a number of equal weights and suspend one of them from each of two points on the bar equally distant from D, and two others from the hook; there will still be equilibrium. Take off the two weights from the bar and hang them one below the other at the point D; with this arrangement also there will be equilibrium. This is an illustration of the fact that two parallel forces, applied at two points in a body, produce the same effect as if they were applied together at the centre of the line which joins the points.

Replace one weight at B, suspend two others at C′, midway between C and D (Fig. 26), and hang three from the hook at A; in this case also there will be equilibrium.

If C'' be taken as the point of support, such that $D\,C'' = \frac{1}{4}\,D\,B$, then it will be necessary when one weight

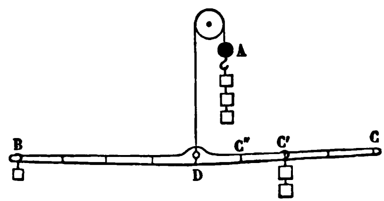


Fig. 26.

is placed at B to suspend four from C'' and five from A.

These experiments, which might be multiplied and varied indefinitely, show that two parallel forces, acting in the same direction, can be counteracted by a single force parallel to them; and consequently they may be replaced by such a force. Moreover, we learn that in magnitude the resultant is equal to the sum of the forces, and if one force be double or triple of the other, the point of application of the first must be twice or thrice nearer the resultant than that of the second; in other words, the distances of the forces from the resultant are inversely proportional to the forces.

38. Parallel forces which act in the same direction may be called *like* forces, and those which act in opposite directions *unlike*.

Composition of Parallel Forces, Geometrical Proof.

39. Let F and F′ be like parallel forces applied at points A and B in a solid body (Fig. 27). We may

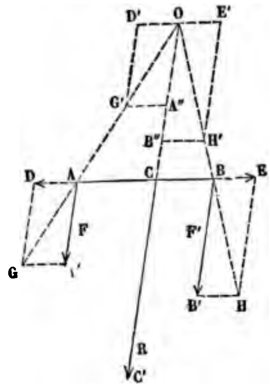


Fig. 27.

apply at the extremity of the rigid line A B two equal and opposite forces f and $-f$ represented by A D and B E, without disturbing the effect of the forces F, F'. The resultant of f and F will be represented by A G, and that of $-f$ and F' by B H. Produce these lines to meet, and let O be the point of intersection. We may transfer the forces represented by A D and A A' to O parallel to themselves, when f will be represented by O D' and F by O A''. Similarly transferring $-f$ and F' parallel to themselves to the point O, $-f$ will be represented by O E', and F' by O B''. Then f and $-f$ neutralise one another, and an effective force remains along O C parallel to the forces F and F', and equal to F + F'. Hence the resultant of two like parallel forces is equal to their sum, and acts parallel to either of the forces in the same direction.

Again, the triangles G A'A, A C O are similar, and therefore their sides are proportional; hence—

$$\frac{GA'}{AA'} = \frac{AC}{CO} \text{ or, } \frac{f}{F} = \frac{AC}{CO} \quad .$$

$$\text{therefore } f \times CO = F \times AC$$

In like manner the triangles H B'B and B C O are similar;

$$\text{hence} \quad \frac{HB'}{BB'} = \frac{BC}{CO} \text{ or, } \frac{f}{F'} = \frac{BC}{CO}$$

$$\text{therefore } f \times CO = F' \times BC$$

$$\text{consequently } F \times AC = F' \times BC$$

Hence the distances CB and CA are inversely proportional to the forces.

Conversely, a given force R may be decomposed into two parallel forces F and F', such that $R = F + F'$ and $F \times AC = F' \times BC$.

By adding $F' \times AC$, or $F \times BC$ to both sides of the above equation, we obtain—

$$\frac{F'}{CA} = \frac{F}{CB} = \frac{F+F'}{CA+CB} = \frac{R}{AB}$$

Hence $F' \cdot AB = R \cdot AC$ and $F \cdot AB = R \cdot CB$.

Again, let E be a point in AB produced (Fig. 28), then by subtracting the equation

$$F \cdot AB = R \cdot CB$$

from the identity $(F + F') \cdot BE = R \cdot BE$ we obtain

$$F \cdot AE + F' \cdot BE = R \cdot CE$$

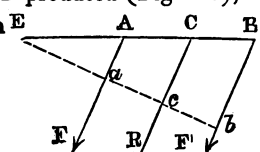


Fig. 28.

Composition of Unlike Parallel Forces.

40. Let F and F' be unlike parallel forces, of which F is the greater (Fig. 29). The force F may be decomposed into two forces; one represented by BB' at B equal and opposite to F', and the other equal to $F - F'$ at a point C, such that,

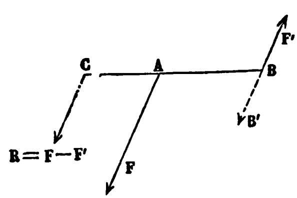


Fig. 29.

$$(F-F')\cdot CA = F'\cdot AB;\ \text{or}\ F\cdot AC = F'\cdot CB$$

The forces at B will be in equilibrium, and an effective force remains at C equal to F — F′. This, then, is the resultant R of the forces F and F′. Therefore the resultant of two unlike parallel forces acting at points A and B is equal to their difference, and acts in the direction of the greater, through a point C, found by producing the line A B on the side of the greater force, so that F : F′ : : CB : CA.

From this proportion we obtain

$$\frac{F}{CB} = \frac{F'}{CA} = \frac{F - F'}{CB - CA} = \frac{R}{AB}.$$

The forces F and F′ are here supposed unequal; two equal and unlike parallel forces form what is termed a *couple*, and have not a single resultant.

To find the magnitude and position of the resultant of any number of parallel forces in the same plane.

41. Let $F_1\ F_2 \ldots F_n$ be the forces, those acting towards one direction being positive, and those in the opposite direction being negative. Let a line A B be drawn so as to intersect the directions of all the forces. Take a point of reference E on A B, and let $d_1, d_2, \ldots d_n$ be the distances of the forces from the point E measured along A B, distances to the right being positive, and those to the left of E negative. Let R_2 be the resultant of the first two forces, R_3 the resultant of the first three, and so on, and let r_2, r_3, be the corresponding distances from E.

By § 39
$$\begin{aligned} F_1 + F_2 &= R_2 \\ R_2 + F_3 &= R_3 \\ &\ldots\ldots \\ R_{n-1} + F_n &= R_n \end{aligned}$$

By adding these equations and remarking that $R_2, R_3, \ldots R_{n-1}$ will occur on both sides the result, and will therefore cancel, we obtain

$$F_1 + F_2 + F_3 \ldots + F_n = R_n \qquad (1)$$

Again by § 39 we also have

$$\begin{aligned} F_1\cdot d_1 + F_2\cdot d_2 &= R_2\cdot r \\ R_2\cdot r_2 + F_3\cdot d_3 &= R_3\cdot r_3 \\ &\ldots\ldots \\ R_{n-1}\cdot r_{n-1} + F_n\cdot d^n &= R_n r_n. \end{aligned}$$

By adding these equations we obtain

$$F_1\cdot d_1 + F_2\cdot d_2 \ldots + F_n\cdot d_n + = R_n\cdot r_n \qquad (2)$$

Let us write $\Sigma F \cdot d$ for the sum of all the products of which $F \cdot d$ is the type, then $\Sigma F \cdot d = R \cdot r = r \cdot \Sigma F$.

Equation (1) determines the magnitude of the resultant, and Equation (2) the point in AB through which its direction passes.

42. *To find the magnitude and position of the resultant of a number of parallel forces which act on a rigid body.*

Let $F_1, F_2, \ldots F_n$, be the forces; it may be shown precisely as above that $R = \Sigma F$. As the forces act on a rigid body we have to consider the positions of the points of application. Let the distances of the points of application from a fixed plane be $p_1, p_2, \ldots p_n$. Let F_1, F_2, act at points A and B. Let a plane through AB, perpendicular to the plane of reference, cut that plane in ab, and let the lines AB, ab, meet in E; then by § 39, $F_1 \cdot AE + F_2 \cdot BE = R_2 \cdot CE$; but if $r_2, r_3, \ldots r_n$, are the distances of the points of application of the successive resultants from the plane of reference,

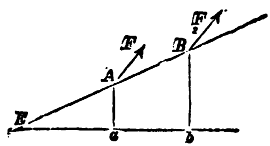


Fig. 30.

$$BE : AE : CE :: p_2 : p_1 : r_2.$$

Therefore $\quad F_1 p_1 + F_2 p_2 = R_2 r_2.$

Similarly $\quad R_2 r_2 + F_3 p_3 = R_3 r_3.$

$$\cdots\cdots\cdots\cdots$$

$$R_{n-1} r_{n-1} + F_n p_n = R_n r_n.$$

By adding, therefore, we obtain

$$F_1 p_1 + F_2 p_2 + \ldots + F_n p_n = R_n r_n$$

$$\therefore \quad \Sigma F \cdot p = r \cdot R = r \cdot \Sigma F.$$

This equation determines r, the distance of the point of application of the resultant from the plane of reference.

If x, y, z, be the values of p for three planes at right angles, and $\bar{x}, \bar{y}, \bar{z}$, the values of r,

$$\Sigma F x = \bar{x} \cdot \Sigma F \quad ; \quad \Sigma F y = \bar{y} \cdot \Sigma F \quad ; \quad \Sigma F z = \bar{z} \cdot \Sigma F.$$

These equations determine $\bar{x}, \bar{y}$, and $\bar{z}$, and therefore the position of the point.

It is important to notice that the equations are not altered by changing the directions of the forces if the points of application are kept in the same relative position; hence

the resultant of a system of parallel forces acting at different points in a rigid body passes through a fixed point the position of which is independent of the direction of the forces. This point is termed the centre of the forces.

43. As an example of the resolution of parallel forces let it be required to decompose a force F applied at A into three others, F′ F″ F‴ applied at three given points B, C, D, in a plane which is not parallel to the direction of the force F (Fig. 31); join B A

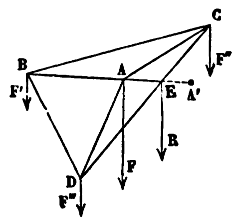


Fig. 31.

and decompose F into F′ at B, and R at E, in the straight lines B A and C D. Now resolve the force R into F″ at C and F‴ at D. Then $F = F' + F'' + F'''$. These forces possess the following remarkable property. Since the triangles A C D, B C D have the same base C D, they are to one another as their heights, or as the lengths A E and B E; hence

$$\frac{AE}{BE} = \frac{ACD}{BCD} \text{ but } \frac{F'}{F} = \frac{AE}{BE} \text{ therefore } \frac{F'}{F} = \frac{ACD}{BCD}$$

The forces F″ and F‴ give similar relations, and therefore

$$\frac{F}{BCD} = \frac{F'}{ACD} = \frac{F''}{ABD} = \frac{F'''}{ABC}$$

If the force F be represented by the area of the triangle B C D the components will be represented by the areas A C D, A B D and A B C.

Exercises.

1. The smaller of two like parallel forces of 24 and 30 lbs. respectively is 5 inches from the resultant, what is the distance between the larger force and the resultant?—*Ans.* 4 in.

2. The resultant of two forces is 66 lbs.; the smaller force is $\frac{10}{22}$ of the resultant and is distant from it 3 ft. 6 in.; find the other force and its distance from the resultant.—*Ans.* 36 lbs., 2 ft. 11 in.

3. The perpendicular distance between two like parallel forces of 1 cwt. and 20 lbs. respectively is 2 ft. 9 in.; find the distance between the smaller force and the resultant.—*Ans.* 28 inches.

4. Two like parallel forces which act at the extremities of a rod $5\frac{1}{3}$ ft. long have a resultant of 1 cwt.; one of the forces is 35 lbs.; find the distances of the points of application of the forces from that of the resultant.—*Ans.* 20 and 44 inches.

5. AB is a rod acted on at A and B by like parallel forces P and Q. C is the point of application of their resultant R. Given that R = 154 lbs., Q = 99 lbs., A C = $5\frac{1}{4}$ ft.; find A B.—*Ans.* 98 in.

6. In the last example, given that A B = 16 in., R = 104, and $\frac{P}{Q} = \frac{3}{5}$; find P, Q, A C, and C B.—*Ans.* 39 lbs., 65 lbs., 10 in., 6 in.

7. If P and Q are unlike parallel forces, and AB a straight line meeting P in A, Q in B, and R in C. Given that P = 15, Q = 21, and A C = 14 in.; find B C.—*Ans.* 10 in.

8. If P = 33 lbs., R = 6 lbs., and A B = 4 in.; find AC.—*Ans.* 18 in.

9. Two like parallel forces act at points in a rigid rod which are respectively 7 and 17 inches from one end; the forces are 13 lbs. and 17 lbs.; find the distance of the point of application of the resultant from the end of the rod.—*Ans.* $12\frac{2}{3}$ in.

10. At points equally distant on a rod 20 inches long, weights of 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs. are suspended. The rod is to be supported by a single thread; at what point must it be

tied that the rod may remain horizontal?—*Ans.* $13\frac{1}{3}$ in. from the end.

11. A weight of $1\frac{1}{2}$ cwt. is carried by two men on a rod 8 ft. long. The weight is hung from the middle; one man is 1 ft. from one end and the other 2 ft. from the other end of the rod; find the weight borne by each.—*Ans.* ·9 cwt. and ·6 cwt.

12. A B C D is a square; forces act along the sides having the directions and proportions as follows:—From A to B, 3 lbs., from B to C, 4 lbs., from D to C, 7 lbs., from A to D, 6 lbs.; find the magnitude and direction of the resultant.—*Ans.* $10\sqrt{2}$ lbs., and acts parallel to the diagonal of the square.

13. A force of 37·5 lbs. acting at a point D in the triangle A B C, is to be resolved into three parallel forces, F_1 at A, F_2 at B, and F_3 at C; find the magnitude of F_3, having given A B = 73, B C = 75, A C = 52, A D = 19, and B D = 60.—*Ans.* 9·5 lbs.

VI.—MOMENTS.

44. Suppose a rod O D, capable of turning about the fixed point O, to be acted on by a force F, the tendency of the force to turn the rod about O depends on the magnitude of the force and on its distance from the point O. We might, for example, double this tendency, either by doubling the force or by keeping the force the same and causing it to act at twice the distance from O. Hence the tendency of the force to turn the rod about O is measured by the product of the force by the perpendicular O D.

The product of any force F *by the perpendicular from a point on its direction is termed the moment of the force with respect to the point.*

Thus if A M represent the force F (Fig. 32), the

moment of F about O is AM × OD. This product is numerically *twice the area of the triangle having the line representing the force for base and the given point for apex.*

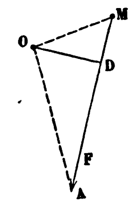


Fig. 32.

We show therefore that the moments of two forces are equal when we prove the equality of the triangles formed by joining the extremities of the lines representing the forces to the given point.

It is evident that the moment of a force about a point in its own direction is zero.

It is convenient to consider moments in one direction, as, for example, that of the hands of a clock as positive, and moments in the opposite direction as negative.

When two forces act on a particle the moments about a point in the direction of the resultant are equal.

45. Let M be the particle and F and F′ the forces (Fig. 33). Complete the parallelogram and take any point D in the resultant, we shall prove that the moments of F and F′ about this point are equal by proving that the triangles AMD, BMD are equal. Now the perpendiculars from A and B on MC are equal, and we may regard MD as the common base; hence the triangles having the same base and equal heights are equal, and therefore the moments of the forces about the point D are equal.

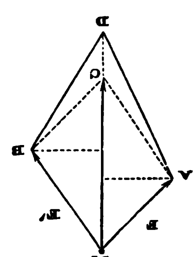


Fig. 33.

The above is a particular case of the following proposition:—

46. *The moment of the resultant of two forces acting on a particle about a point in their plane is equal to the algebraical sum of the moments of the forces.*

Let MB and MA represent the forces and MC their resultant.

1st. Let the point O be without the parallelogram AMBC (Fig. 34). Join the point O with the angular points of the parallelogram and also project these points on any line KL perpendicular to MO. Now the moment of the resultant about O is twice the triangle OMC, and the moments of the forces twice the triangles OMB and OMA. All the moments are in the same direction, hence we shall prove the proposition when we show that

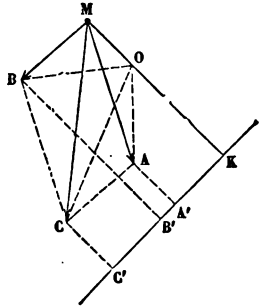


Fig. 34.

$OMC = OMB + OMA$. The side OM may be taken as the common base of all the triangles, and then the heights are respectively KC', KB', and KA'. Since BC and MA are equal and parallel their projections are equal, and therefore $KA' = C'B'$.

$$\text{Now} \quad KC' = KB' + C'B'$$
$$= KB' + KA'$$

Multiply by OM

therefore $KC' \times OM = KB' \times OM + KA' \times OM$, and these products are twice the areas of the triangles;

hence the moment of the resultant equals the sum of the moments of the forces.

2nd. Let the point O be within the parallelogram (Fig. 35).

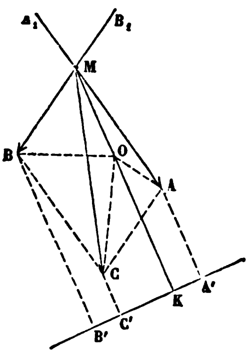


Fig. 35.

In this case the moments of the resultant and one force are opposite to the moment of the other force; hence we have to prove that the triangle MOC = triangle MOB — triangle MOA.

As before $$B'C' = KA'$$
$$KC' = KB' - B'C'$$
$$= KB' - KA'$$
therefore $KC' \times OM = KB' \quad OM - KA' \times OM.$

Hence the proposition is fully proved.

The preceding proposition is also true when the forces do not meet, thus :—

47. *The algebraical sum of the moments of two parallel forces about any point in their plane is equal to the moment of their resultant.*

Let F F′ be like parallel forces, and R their resultant, (Fig. 28); then we have seen that if E B be any line meeting the forces in the points A, B, C, F·AE + F′ BE = R·EC; and since every line from E cutting the forces will be divided proportionally to E B, the perpendicular E b is so divided; hence we obtain.

$$F \cdot a E + F' \cdot b E = R \cdot c E$$

In like manner the proposition may be proved for unlike parallel forces, and may be extended to any number of forces.

It follows that when any number of forces are in equilibrium, the algebraical sum of the moments about any point is zero; when the forces have a resultant, the sum of the moments about a point in the resultant is zero; when the forces are equivalent to a couple, the moment never vanishes but is the same for every point in their plane.

When several of the forces acting on a body are unknown, we can frequently find an equation involving only one of the unknowns by taking moments about some point through which this force does not pass.

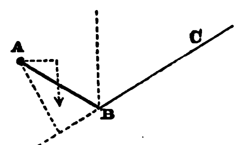
Example. A beam A B has one end attached to a hinge A, and the other end attached to a cord B C, one end of which is tied to a peg. The weight of the beam is 50 lbs., and may be supposed to act at its middle point. The beam and cord make angles of 60° on opposite sides of the vertical. Find the tension of the cord.

Here then are two unknown forces, the reaction of the peg R, and the tension of the cord T. If, however, we take moments about the hinge, we shall obtain an equation not involving R, for the moment of R about *this point is zero.*

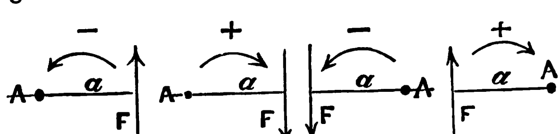
Let $2\,l$ = the length of the beam; draw a vertical line through the middle point of A B to represent the direction of the weight. The perpendicular on this line from A is $\frac{\sqrt{3}}{2}\cdot l$; hence the moment of the weight about A is $25\sqrt{3}\cdot l$.

The perpendicular on the direction of the cord is $l\sqrt{3}$ and therefore the moment of the tension is $T\cdot l\sqrt{3}$.

$$\text{Therefore } T\cdot l = 25\cdot l$$
$$\text{and } T = 25.$$



To determine the proper sign of each of the moments imagine the line drawn from A to the force to be a rod jointed to the fixed point A, then if the force would tend to turn this rod in the direction of the hands of a clock make the sign of the product +, but if the force would tend to turn the rod in the opposite direction, then make the product —, as in the following figure:—



Exercises.

1. Two equal forces act at an angle of 60°; O is a point such that the perpendiculars from it on the directions of the forces are

respectively 5 in. and 2 in.; find the distance of the resultant from the same point.—*Ans.* $\sqrt{3}$, or $2\frac{1}{3} \cdot \sqrt{3}$.

2. Three forces, P, Q, and R, acting on a point are in equilibrium; the ratio of the moments of R and Q about a point in the line bisecting the angle between them is $\frac{4}{5} : \frac{P}{R} = \frac{3}{5}$ and R = 20 lbs. Find P and Q.—*Ans.* 12 lbs. and 15 lbs.

3. Forces of 10 lbs. each act in order along the sides of a regular hexagon; find the sum of the moments about one of the angular points, each side being 1 foot in length.

4. A B C is an isosceles triangle, and D any point in the base B C. If equal forces act along the sides A B, C A, prove that the sum of the moments about D is independent of the position of D.

5. Three equal forces act in order along the sides of an equilateral triangle; show that the sum of the moments about any point within the triangle is invariable.

6. If the algebraical sum of the moments of two forces about two points be the same, prove that the line joining these points is parallel to the resultant.

7. A circular disc, of radius $\sqrt{130}$ inches, in a vertical plane is movable about an axis through the centre. From the extremities of two radii at right angles to each other, weights of 14 and 18 lbs. respectively are suspended; find the depths of the points of suspension below the horizontal line through the centre.—*Ans.* 9 and 7 inches.

8. In pulling a weight along the ground by a rope inclined to the horizon at an angle of 45° a power of 40 lbs. is exerted; what force applied horizontally would drag the body?—*Ans.* 28·28.

9. Two forces which are to each other as 2 to $\sqrt{3}$ act upon a point, and produce a force equal to half the greater; find their inclination.—*Ans.* 150°.

10. A thread 12 feet long is fastened at points A and B in the same horizontal line 8 feet apart. At C and D, points 4 feet and 5 feet respectively from A and B, weights are attached; what must be the ratio of the weights that C D may be horizontal?—*Ans. 17 to 8.*

VII.—REACTION OF SURFACES.

48. When a body is kept at rest under the action of several forces, it frequently happens that the reaction of one or more surfaces assists in preserving equilibrium. This reaction acts in a direction perpendicular to the surface. Thus, when a sphere rests on a horizontal plane the reaction of the plane is evidently equal and opposite to the weight. When the surface of the body is in contact with the plane in many points the reaction of the plane is *distributed* over these points. We cannot generally find out the pressures at the particular points, but the resultant of these pressures must act vertically upwards through the base of the body. Again, suppose the body pressed against the surface so as to be at rest, the various points in contact present reactions which have a resultant equal and opposite to the resultant of the forces pressing the body against the surface.

49. Sometimes the reaction of the surface is such as to permit displacement only in certain directions. The body is then said to be *constrained*. Suppose, for example, a ring M to be supported on a metallic rod bent into the form of an arc A B (Fig 36). If a force F be applied at the point M perpendicularly to the curve, the ring will not be moved, for the force makes equal angles with the two directions in which the point might be displaced, and *there is no reason* why it should move in one more than

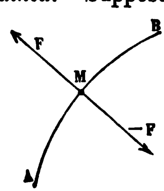


Fig. 36.

in the other direction. The reaction of the curve must therefore be equal to — F.

Suppose now a force F applied at M in another direction (Fig. 37); this force may be decomposed into two, one T along the tangent to the curve, and the other N along the normal (*i.e.* the perpendicular to the tangent). The component T tends to displace the point, and the part N is counteracted by a reaction — N equal and opposite to it.

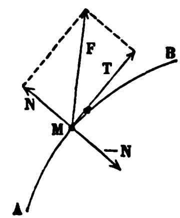


Fig. 37.

50. Consider now a rigid body having one point in it fixed (Fig. 38); the body can turn only about this point. Let a force F act at another point in the body, if the line of direction of this force pass through the fixed point, we may suppose the force to act at that point, and it will then be in equilibrium with the resistance. If the line of direction of the force do not pass through the fixed point, the body will evidently turn about that point.

51. If a body have two points, A B, fixed, the line joining them is fixed, and the body is capable of moving only about this line. Suppose a force F applied at a point M in the body so that the axis A B and the direction of the force are in the same plane (Fig. 39).

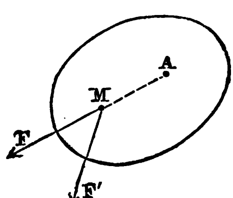


Fig. 38.

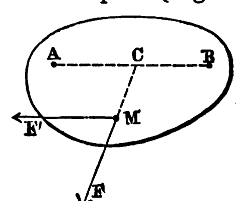


Fig. 39.

If the line of direction of the force meet the axis in the point C we may transfer the point of application to C, and the force will there be counteracted by the reaction of the fixed axis. If the force be parallel to the axis it tends to turn the body in the plane A B M, but this tendency is evidently counteracted by the reaction of the fixed points. If a force be applied at M perpendicular to the plane A B M, it has a moment about a point on the axis not counteracted by the reactions at A and B.

If the force be neither in the plane A B M nor perpendicular to it, suppose a plane containing the line of direction of the force and perpendicular to the plane A B M to intersect the latter in the line C M, we may replace the force by two others in the same plane, one in the direction of C M, the other perpendicular to it, the first being counteracted by the reaction of the axis, the second tending to turn the body about the axis. The product of the latter component and the perpendicular distance of M from the axis is termed the moment of the force about the line A B.

52. The moment of a force with respect to a line may be thus defined :—

If the force be resolved into two components, one in the same plane as the line and point of application of the force and the other perpendicular to this plane, the product of the latter component by the perpendicular distance of the point of application of the force from the line, is the moment *of the force about the line.*

It is evident that if the point of application of a force be moved parallel to the fixed line, and the force *parallel to itself*, the moment remains the same.

Hence, if a body having a fixed axis be acted on by a system of forces, and the points of application of the forces be projected on a plane perpendicular to the axis, we may suppose the forces to act at these points without disturbing the effect of the forces. We may then conclude *that the algebraical sum of the moments of the forces about the axis is equal to the moment of the resultant*; *and if the algebraical sum of the moments vanish, the forces are in equilibrium.*

Exercises.

1. A cord is tied to the end of a smooth rod, inclined at an angle of 45° to the vertical, and supports a ring weighing 3 lbs. through which the rod passes; find the pressure on the rod, and the tension of the cord.—*Ans.* $\frac{3}{\sqrt{2}}$ lbs.

2. A beam, A B, rests with one end, A, against a smooth vertical wall, and the other end, B, on a smooth horizontal plane; it is prevented from sliding by a cord tied to one end of the beam and to a peg at the bottom of the wall; the length of the beam is 10·6 feet, and the length of the string 9 feet. Suppose the weight of the beam to be 112 lbs. and to act vertically through its middle point, and show that reaction of wall = tension of string = 90 lbs.; reaction of plane = weight of beam.

3. Show from article 51 that if three forces acting at different points in a rigid body keep it at rest they must lie in the same plane.

4. Explain the action of the rudder of a boat.

5. Three smooth vertical posts are fixed at the angles of an equilateral triangle, and a cord is passed round them, to each end of which a force of 100 lbs. is applied; find the pressure on each post.—*Ans.* $100 \cdot \sqrt{3}$.

6. If the posts form a square and the cord passes twice round the square, find the pressure on each.—*Ans.* $200 \cdot \sqrt{2}$.

7. A sailing vessel moves in a direction making an acute angle

with that of the wind; explain the action of the forces which produce this result.

8. Explain the action of the forces by which a kite is supported in the air.

9. A ladder, the weight of which may be regarded as a force acting at a point one-third the length from the foot, rests with one end against a peg in a smooth horizontal plane, and the other on a wall. The point of contact with the wall divides the ladder into parts which are as 1 : 4; having given that the ladder weighs 120 lbs., and makes an angle of 45° with the horizontal plane, find the pressure on the peg.—*Ans.* 25 lbs.

(Take moments about the top of the wall. Then if $P =$ pressure on the peg, and $R =$ the reaction on the horizontal plane, we obtain $R - P = 70$. Now resolve the forces in the direction of the ladder, therefore $P + R = W = 120$.)

10. In the above case find the reaction of the wall.—*Ans.* $25 \cdot \sqrt{2}$.

(Take moments about the foot of the ladder.)

11. If a weight of 10 lbs. be hung at the smaller end of the ladder, what will be the whole pressure on the horizontal plane?

12. A boat which is under the action of a S.E. wind and S. current is attached by a cord to a fixed point. The tension of the cord is 150 lbs. and its direction 20° from the south. Show by a construction how to find the forces of the wind and current. —*Ans.* 72½ lbs.; 192 lbs.

13. If a straight rod be balanced on a point, and weights of 1, 2, and 3 lbs. be suspended at distances of 6, 12, and 18 inches from the point in one direction, and 2, 3, 4 lbs. at distances of 4, 10, and 12 inches in the other, find where a weight of 1 lb. must be placed so as to keep the rod at rest.—*Ans.* 2 in. from the point.

14. A cord, C B, has one end, C, attached to a point in a vertical wall, and the other end, B, to the extremity of a beam, A B, which rests against the wall. If A C = 5 ft. 5 in., A B = 8 ft. 1 in., and weighs 130 lbs., find the length of C B, neglecting the friction between the wall and beam.—*Ans.* 148·6 inches.

15. A sphere weighing 200 lbs. rests between two planes in-

clined to the horizon at angles respectively of 30° and 60°; find the pressures on the planes.—*Ans.* 100 lbs. and 173·205 lbs.

16. A beam, A B, is placed with one end, A, inside a hemispherical bowl, and a point, C, in it, resting on the edge of the bowl; show, by a construction, how to find the inclination of the beam, friction being neglected.

17. If the radius of the bowl be 10 ft., and the beam make an angle of 30° with the horizon, find the length of the beam.—*Ans.* $13\frac{1}{3}\cdot\sqrt{3}$ or 23·094.

18. A B C is a triangle capable of moving about the right angle, B, in a vertical plane; find the ratio of the weights which must be attached to A and C, that the side A C may be horizontal, having given that A B = 6·15 and A C = 9·53 inches.—*Ans.* 615^2 to 728^2.

VIII.—CENTRE OF GRAVITY.

53. The attraction of the earth which causes a body to have weight, acts on every particle of the body; thus, if we take a stone and pound it into small fragments the sum of the weights of the small particles will be equal to that of the whole stone.

If one of these particles be attached by a fine thread to a fixed point o, the thread will take the direction of the vertical through o. If several of them be suspended from points near together, the threads will be parallel. When therefore the particles are united so as to form the body, we may regard their weights as a system of parallel forces.

Suspend the body by a point A (Fig. 40), the resultant of the weights of the particles will be equal to

their sum, and will have the direction of the vertical through A. Suspend the body again from another point; the weight of each particle will have the same magnitude and the same point of application as before, but the direction of the forces with regard to the body will be changed. The result is the same as if each force had been turned about its point of application; hence the new line of support or direction of the resultant will intersect the old one in the centre of the forces. If the body were composed of a plastic material, and pierced in the direction of the line of support in several different positions, all the lines of perforation would intersect in a common point. This point is termed the centre of gravity of the body. We are led therefore to the following definition:—

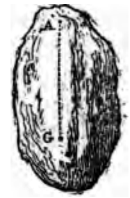


Fig. 40.

54. *The resultant of a system of parallel forces acting on a rigid body passes through a fixed point the position of which is independent of the direction of the forces. If the forces be the weights of the several elements of the body, the fixed point is termed the centre of gravity.*

When a body is suspended from a point A it comports itself as if its weight were concentrated at the centre of gravity G. We may consider therefore that two forces act upon it—the resultant weight along the vertical through G and the reaction of the point of support along the vertical through A. When A and G are in the same vertical line the body is at rest; hence, if the C. G. be supported, the whole body will be supported.

We will now consider the position of the centre of gravity in certain figures.

The process of finding the centre of gravity of a number of isolated particles is precisely the same as that of finding the centre of a system of parallel forces.

We shall be concerned only with bodies through the volumes of which matter is distributed uniformly. Such bodies are termed homogeneous. A solid body is homogeneous when any two parts of equal volume are exactly of the same weight. The determination of the centre of gravity of a homogeneous body is therefore a purely geometrical question. The weights of different portions will be proportional to their volumes; hence we may treat the volumes as the forces.

Again, consider a very thin sheet of metal, or paper, of uniform thickness. The weights of any two portions will be proportional to the areas; hence we may treat the areas as forces, and seek the centre of gravity of the *surface*.

In like manner if we take a very thin wire of uniform thickness, the weights of different portions will be proportional to their lengths. We may therefore find the centre of gravity of a heavy line.

55. The following considerations will assist us in solving problems connected with the centre of gravity.

1. If a body be symmetrical about a plane, the C. G. lies in that plane. Every particle on one side corresponds to an equal particle on the other. Hence the C. G. of every pair of particles lies in the plane, and, therefore, so also does the C. G. of the whole.

2. It follows that if a body have two planes of symmetry, the C. G. lies in this line of intersection; and if it have three planes of symmetry intersecting in two lines, the C. G. is at the point where the lines cut one another.

3. If an area be symmetrical about a line, the C. G. lies in that line.

4. If a body have a centre of figure, that is, a point such that all lines drawn through it to the outline of the figure are bisected in the point, the centre of figure is the C. G.

For any two lines drawn through this point will contain figures on opposite sides of the vertex, in every respect equal. If any point in one of these figures be joined with the corresponding point of the other, the line drawn will be bisected by the centre, therefore the line joining their C. G.s will be bisected by the centre of figure. Consequently, this point is the C. G. of the two figures; similarly it is the C. G. of every other pair included by lines through the centre, and therefore it is the C. G. of the whole.

From these facts we may conclude at once that—

1. The C. G. of a straight line is its middle point.
2. The C. G. of the circumference or area of a circle is the centre.
3. The C. G. of the perimeter or area of a parallelogram is the point of intersection of the two diagonals, for this point is the centre of figure.
4. The C. G. of the volume or surface of a sphere is the centre.
5. The C. G. of a right circular cylinder is the middle point of the axis.
6. The C. G. of a parallelopiped is the point of intersection of two diagonals.

To find the C. G. of the Surface of a Triangle.

56. Let A B C be the triangle (Fig. 41). The C. G. may be found experimentally thus: — Suspend it by one of the angles, A, from a point O, and mark on the triangle the line A D *in the direction* of the

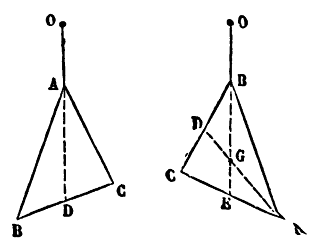


Fig. 41.

plumb-line through o. Take off the triangle and suspend it by another point B. Draw B E on the direction of the plumb-line, intersecting A D in a point G. The point is evidently the C. G. If the triangle be again suspended, the line of support will pass through G. It will be found that E and D bisect the sides A C and D C. Hence to find the C. G. of a triangle, join the middle points of two sides with the opposite angles.

To prove the above Proposition Geometrically.

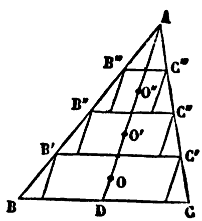


Fig. 42.

57. Join A with the middle point D of the base (Fig. 42). Take any number of points B′ B″ B‴ in A B, and through them draw parallels to B C and A D, so as to form a series of parallelograms. A D divides the parallels B C, B′C′, &c., in the same proportion; since it bisects one of them, B C, it bisects all the others, and hence the centres of all the parallelograms lie on the line A D. This will be the case whatever may be the number of the parallelograms. Their sum is always smaller than the triangle, but approaches nearer and nearer to it as the number of divisions in A B is increased. Since the C. G. of the sum of the parallelograms lies in A D, therefore the C. G. of the triangle is in A D. Similarly the C. G. lies in the line B E drawn from B to the middle point of A C. It must therefore be at G, the point of intersection of the two lines, A D and B E.

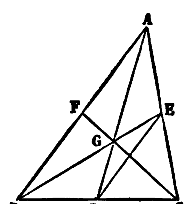


Fig. 43.

Join D E. Since the triangles, C E D, C A B are similar, $\frac{CD}{CB} = \frac{DE}{AB}$; and since the triangle D G E is similar to A G B

$$\frac{DG}{AG} = \frac{DE}{AB} \text{ therefore } \frac{DG}{AG} = \frac{CD}{CB} = \frac{1}{2}.$$

Therefore DG $= \frac{1}{2}$ AG or $\frac{1}{3}$ AD.

Hence the C. G. of a triangle is in the line joining the apex with the middle of the base, and at one-third of its length from the base.

To find the C. G. of three equal Particles placed at the angles A B C of a Triangle.

58. The resultant of P at B and P at C will be 2 P at D (Fig. 43), and the C. G. of 2 P at D, and P at A is at a point G, such that D G equals $\frac{1}{2}$ A G. Hence the C. G. of the three equal particles coincides with the C. G. of the triangle.

To find the C. G. of the Perimeter of a Triangle.

59. The C. G. of each side is at its middle point. Hence we have to find the C. G. of forces acting at the middle points A′ B′ C′ of each side, and being proportional to the lengths of the sides (Fig. 44). Join the points A′ B′ C′. The C. G. of the forces at C′ and B′ is at a point D, such that

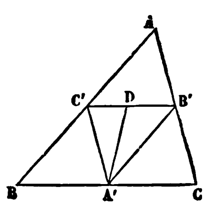


Fig. 44.

$$\frac{DB'}{DC'} = \frac{AB}{AC} = \frac{\frac{1}{2}AB}{\frac{1}{2}AC} = \frac{A'B'}{A'C'}.$$

Hence the C. G. of the whole perimeter is in the line A′D. Since the line A′D divides the base C′B′ into parts proportional to the sides, it bisects the angle A′. Similarly we may show that the C. G. lies in the line bisecting the angle B′. Now the intersection of the bisectors of two angles of a triangle is the centre of the inscribed circle; hence the C. G. of the perimeter of a triangle coincides with the centre of the circle inscribed in the triangle, whose angular points bisect the sides of the original triangle.

To find the C. G. of a Triangular Prism.

60. Suppose one of the triangular bases A B C divided into parallelograms as in Fig. 42, and imagine planes to pass through their sides parallel to the edges of the prism (Fig. 45), these planes will form a series of

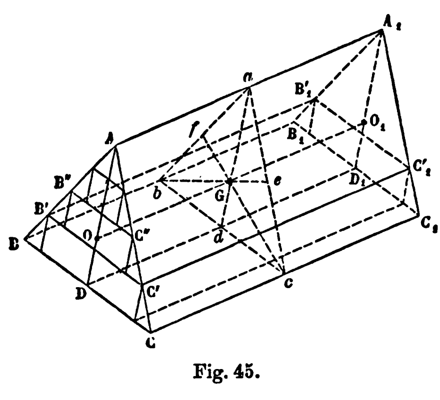


Fig. 45.

parallelopipeds within the prism. The C. G. of each will be in the plane A′ A D which bisects the sides parallel

to B C, and in the plane *a b c* which bisects the lateral edges of the prism. The C. G. of the whole is therefore in the line of intersection *a d*, whatever be the number of the parallelopipeds; hence the C. G. of the prism will be in this line.

Similarly it may be shown that the C. G. of the prism is in *b e*; it is therefore at G, the C. G. of the triangle *a b c*. The point G is evidently the middle point of the line O O′, which joins the C. G.s of the two ends.

The same rule applies to any prism whatever, for such a prism may be decomposed into triangular prisms,

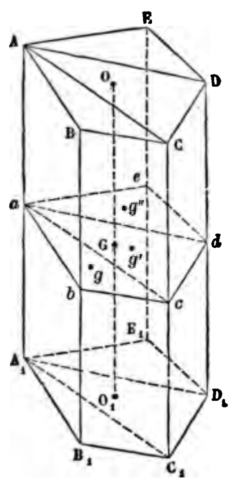


Fig. 46.

all the C. G.s of which lie in a plane *a b c d e*, parallel to the bases, and cutting the line joining the C. G.s of *the two ends* in the middle point G (Fig. 46).

To find the C. G. of a Triangular Pyramid.

61. Let SABC be the pyramid. Find D the C. G. of the base, and join SD. Suppose a number of planes drawn parallel to the base and intersecting the other faces of the pyramid, and also other planes through the lines of intersection parallel to SD. These planes enclose a series of prisms, all the C.G.s of which lie on

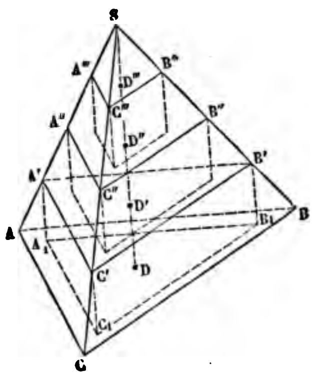


Fig. 47.

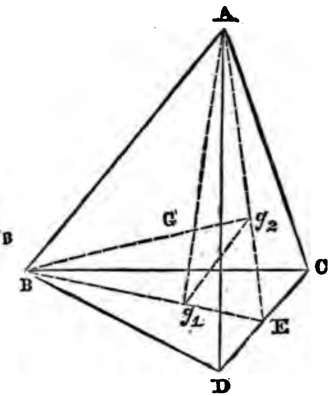


Fig. 48.

the line SD. Hence whatever may be the number of prisms, the C. G. of their sum is in the line SD, and therefore the C. G. of the pyramid to which their sum approaches lies in the same line. Similarly, if E be the middle point of the edge DC of a pyramid ABCD and $g_1\, g_2$ the C. G.s of two adjacent faces (Fig. 48), the C. G. of the pyramid lies in both the lines $A g_1$ $B g_2$, and is therefore at their point of intersection G. Now the triangles ABE, $E g_1 g_2$ are similar, hence $\frac{g_1 g_2}{AB} = \frac{g_1 E}{BE}$

$= \frac{1}{3}$; also the triangles ABG, $g_1 g_2 G$ are similar,

therefore $\frac{g_1 g_2}{AB} = \frac{g_1 G}{AG}$. Hence $g_1 G = \frac{1}{3}$ of AG $= \frac{1}{4}$ of A g_1. *The C. G. of the pyramid is therefore in the line joining the C. G. of the base and the apex, and at one-fourth of its length from the base.*

It will be seen that the C. G. of the pyramid coincides with the C. G. of equal weights placed at the four corners.

To find the C. G. of a Pyramid with Polygonal Base.

62. Decompose the solid into triangular pyramids by planes passing through one edge S A. Take a point a in S A such that A $a = \frac{1}{4}$ S A. A plane through this point parallel to the base will cut the pyramid in a section similar to the base, and will divide all lines from S to the base in the same proportion. Hence the C. G. of each pyramid coincides with the C. G. of the triangular section. Therefore the C. G. of the whole pyramid lies in this plane and coincides with the C. G. of the whole section.

Now the C. G.s of the similar sections will be similarly situated; hence if G be the C. G. of the base the line S G will intersect the plane of the section in G, its centre of gravity. *Hence the C. G. of every pyramid is in the line joining the C. G. of the base with the apex at one-fourth of its length from the base.*

We may consider a cone as a pyramid with a great number of faces; hence the C. G. of a cone is in the line joining the C. G. of the base with the apex at one-fourth of its length from the base.

To find the C. G. of the Surface of a Pyramid.

63. *If the point* a be taken, so that 2 A a = S a,

and *a b* be drawn parallel to A B, a plane through *a b* parallel to the base will cut all lines from S to the base in the same proportion, and will therefore contain the C. G.s of all the triangular faces. If the pyramid be regular it will be symmetrical about planes intersecting in the line joining the centre of the base and the apex. Hence this line will contain the C. G. of the surface at one-third of its length from the base.

It follows that the C. G. of the convex surface of a right circular cone is in the line joining the centre of the base with the apex at one-third its length from the base.

Properties of the Centre of Gravity.

64. We have seen that when a body is suspended from a point the vertical through that point must contain the C. G. of the body. The same is true when the body rests on a point.

If the body rest on more than one point in the same horizontal plane the resistances at these points will be *like* parallel forces and will therefore have a single resultant parallel to them. The direction of this resultant will lie within the base formed by the points; hence *the vertical through the C. G. of the body must fall within the base.*

The force required to move a body may vary with the position of the body. Let A B C D (Fig. 49) be the section of a prism resting on a horizontal plane. Turn it about one edge A. The C. G. describes a circle, and the force required to move it decreases as the C. G. ascends; in other words, the stability of the body

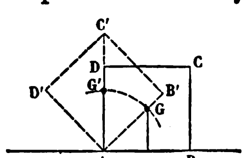


Fig. 49.

decreases as the C. G. is raised. When the C. G. arrives at the position G′ in the vertical through A, the body reaches the limit of stability. In this position the equilibrium is mathematically possible, for the resistance of the surface through A will be in the same straight line as the C. G. The slightest force will be sufficient to move the body, and when disturbed it will fall away from this position.

65. When a body in equilibrium would return to its original position if slightly displaced, the equilibrium is said to be *stable*; when the body would fall away from its first position if slightly displaced, the equilibrium is said to be *unstable*.

Remark that there is equilibrium only when the C. G. occupies the lowest or highest possible position, and the equilibrium is stable in the first position and unstable in the second. Hence every body suspended from a point is in stable equilibrium, and every body supported above a point or line is in unstable equilibrium.

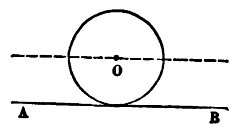


Fig. 50.

Consider now a sphere resting on a horizontal plane, the reaction of the plane and the weight of the body will act along the same straight line in every position. If therefore the sphere be slightly displaced it will have no tendency either to return to or move away from its first position. Its C. G. will neither be raised nor lowered. The *equilibrium in* this case is *neutral.* When the body rests on

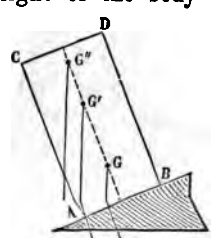


Fig. 51.

an inclined plane (Fig. 51), it is still necessary that the vertical through its C. G. shall fall within the base. Let C D represent a prism, and A B its base on an inclined plane; the resultant of the resistance offered by the points in contact perpendicular to the plane will evidently pass through a point in A B. If the body be prevented from sliding by the roughness of the plane, the perpendicular reaction of the plane and the force caused by this roughness meet in a point in A B, and have therefore a resultant through the same point. That this resultant may be in equilibrium with the weight, the vertical through the C. G. of the prism must evidently fall within A B. If the C. G. be at G, the equilibrium will be stable, if at G′ unstable, and if at G″ the body will fall.

66. The necessity of keeping the C. G. so that the vertical through it shall fall within the base formed by the points of support has constantly to be considered. When a man stands or walks, the line of direction of his weight must fall within the base formed by his feet. If he take a load in one hand, the C. G. of the whole figure will be thrown towards the load, and he naturally raises the free arm and leans towards the other side, thus making the new C. G. fall over the base formed by his feet. If he carry the burden on his back, he bends forward; if he bear it in front, he leans back. In every case the load and the *bearer form a whole* and the C. G. of the whole is displaced on

Fig. 52.

the side of the load so that it is necessary to bring the new C. G. over the base of support.

Consider now a cart with two wheels; if the vertical through the C. G. of the loaded cart fall in the line joining the points of contact of the wheels and ground, the weight is supported by the road; in descending an incline the vertical through the C. G. will fall in front of this line, and part of the weight must be borne by the horse; in an ascent the C. G. falls behind, and the load tends to lift the horse. When the cart is properly loaded on a horizontal road, the vertical through the C. G. falls near the middle of the line between the wheels, but on a road inclined towards one side the vertical falls nearer the lower wheel. So long as it does not pass the lowest point of this wheel equilibrium is maintained, but as soon as it passes it the cart is overturned The higher the C. G. is above the ground the sooner is this position reached.

Fig. 53.

To find, geometrically, the C. G. of any rectilineal figure.

67. Consider first a quadrilateral, A B C D (Fig. 54). Join A C, and find G_1, G_2, the C. G.s of the triangles A B C, A D C. The C. G. of the whole must be in the line G_1 G_2. Now join B D, find G_3 G_4 of the triangles B A D, B C D: the C. G. of the whole must be in the line G_3 G_4. It is therefore at *their point of intersection*.

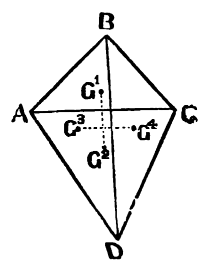


Fig. 54.

If the figure have five sides we can divide it in two ways into a triangle and a quadrilateral, and join their C. G.s.

From a pentagon we may proceed to a hexagon, and so on.

To find the C. G. of a part of a body.

68. When we know the volume and position of the C. G. of the whole body, and the volume and position of the C. G. of a part cut off, we can find the C. G. of the remainder.

Let V be the volume of the whole, V_1 of the part cut off, then $V - V_1$ is the volume of the remainder. Let a be the distance between the C. G. of the whole and that of the part cut off, and let x be the distance between the C. G. of the remainder and that of the whole; then V may be considered as the resultant of parallel forces V and $(V - V_1)$ at distances a and x from it.

$$\text{Therefore } V_1\, a = x \cdot (V - V_1)$$

$$\text{Hence } x = \frac{V_1 \cdot a}{V - V_1}$$

As an example, let it be required to find the C. G. of the trapezoid ABDC formed by cutting off a triangle EAB from a triangle ECD by a line AB through the middle points of the sides CE, DE.

Let M be the middle point of CD, and let A = the area of ECD, then EAB $= \frac{1}{4} \cdot$ A, and ABDC $= \frac{3}{4}$ A. Let EM $= a$, then the C. G. of the whole is $\frac{2}{3}\, a$ from E, and the C. G. of the part cut off $\frac{1}{3}\, a$ from E; therefore the distance between them is $\frac{1}{3}\, a$.

Hence $\frac{A}{4} \times \frac{a}{3} = \frac{3}{4} A \times x$ and $x = \frac{a}{9}$

69. We may, however, take the distance of the C. G.s from a fixed line, or a fixed plane, and form the equation thus:—Let $a =$ the distance of the C. G. of the whole from the line, a_1 the distance of the C. G. of the part cut off, x that of the part left,

$$\text{then } V \times a = (V - V_1)\, x + V_1 \times a_1$$

$$\text{or, } x = \frac{V \cdot a - V_1 a_1}{V - V_1}$$

For example, find the C. G. of a conical shell contained between two right circular conical surfaces, having the same axis, the outer diameter of the shell being 8 in., the inner diameter 6 in., and the height of the whole 12 in. Since the heights are proportional to the diameters of the bases, the height of the cavity is to 12 as 6 to 8; hence it is equal to 9 inches.

The volume of a cone the height of which is h, and radius of base $r = \pi \cdot r^2 \frac{h}{3}$.

Hence the volume of the solid contained by the outer conical surface $= \pi \cdot 64$, and that contained by the inner surface $= \pi \cdot 27$, therefore the volume of the shell $= \pi \cdot 37$. The C. G. of the whole cone is 3 in. from the base: that of cavity $2\frac{1}{4}$ in. from the base.

$$\text{Hence } \pi \cdot 64 \times 3 = \pi \cdot 27 \times 2\tfrac{1}{4} + \pi \cdot 37 \times x$$

And $x = 3\frac{81}{148} =$ the distance of the C. G. of the shell from the base.

EXERCISES.

1. Find the C. G. of two solid bodies of 12 lbs. and 20 lbs. respectively, the line joining their centres of gravity being 2 ft. 8 in.

2. The distance of the C. G. of two heavy particles from the greater is 5 inches, the particles are respectively 60 and 72 grammes; find the distance between them.—*Ans.* 11 in.

3. Weights of 2 lbs., 4 lbs., 6 lbs., and 8 lbs. are placed so that their C. G.s are in a straight line, and 6 inches apart; find the distance of their common centre of gravity from that of the larger weight.—*Ans.* 6 in.

4. Find the C. G. of the remainder of a square when one of the triangles into which the diagonals divide it is taken away. (§ 58)—*Ans.* If $a =$ the side of the square the C. G. is $\frac{a}{9}$ from the centre.

5. A B C D is a quadrilateral in which the base A D is double each of the other sides; find the C. G. (§ 58)—*Ans.* $\frac{4}{9}$ths of the height from the base.

6. If three men support a triangular board at its three corners, what portion of the weight will they bear?

7. A figure is formed of a square and an isosceles triangle, equal in area to half the square on one of the sides; find the distance of the C. G. of the whole from the common side.—*Ans.* $\frac{2}{9}$ths of the side.

8. The middle point of one side of a square is joined with the middle points of the adjacent sides, and the triangles thus formed cut off; find the C. G. of the remainder.—*Ans.* If a be the side, the C. G. is $\frac{a}{9}$ from the centre (§ 58).

9. A mass of granite, 20 ft. long, 4 ft. wide, and $2\frac{1}{4}$ ft. high, rests on three props of the same height, two being placed at the corners of one end, and the other at the middle point of the *opposite end* of the base. Having given that a cubic foot of

granite weighs 2625 oz., find the pressure on each prop.—*Ans.* 236250 and 118125 oz.

10. Two spheres touch one another, find the distance of the C. G. from the point of contact, the radii being respectively 8 in. and 12 in.* —*Ans.* $7\frac{3}{7}$ in.

11. Find geometrically the C. G. of weights of 3, 4, 5, and 6 lbs. placed at the corners of a square, the side of which is 1 ft.

12. Find the C. G. of two blocks of marble of the same uniform density, the lower one being 4 ft. long, 4 ft. broad, and 2 ft. thick, and the upper 8 ft. long, 2 ft. broad, and 2 ft. thick, and an edge of the upper lying along an edge of the lower, so that the centres of the two edges coincide.—*Ans.* The C. G. is 6 in. from the centre of the upper face of the lower block.

13. A ladder 25 ft. long, weighing 60 lbs., and having the C. G. 5 ft. from one end, is carried by two men at the ends; what weight is borne by each?—*Ans.* 48 lbs. and 12 lbs.

14. Find the C. G. of a uniform circular disc out of which another circular disc has been cut, the diameter of the latter being a radius of the former.

15. Find, geometrically, the C. G. of a heavy bar, 10 ft. long, bent so as to form an angle 4 ft. from one end.

16. If A B C be a right-angled triangle, p, q and r weights at the angles, find the distance of the C. G. of the weights from the angle A, having given A B = 80 in., B C = 39 in., C A = 89 in., p = 15 lbs., q = 14 lbs., r = 12 lbs.—*Ans.* 52 in.

(Find first the C. G. of q and r and its distance from A.)

17. If two triangles stand on the same base, the line joining their centres of gravity is parallel to the line joining their vertices.

18. A circular table, weighing 168 lbs., is supported on four legs in the circumference which form a square; find the least weight which being at the circumference will overturn the table —*Ans.* 406 lbs.

19. Three equal weights are placed at distances of 120° on the

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* The vol. of a sphere $= \frac{4}{3}\pi r^3$ ($\frac{22}{7}$ is an approximation to π), and *the vols. of spheres* are as the cubes of their radii.

circumference of a circular table; what must be the magnitude and position of a fourth weight which will bring the C. G. of the whole to the centre of the table?

20. A disc of lead, 20 inches in radius and 2 inches thick, is laid on another disc 40 inches in radius and 3 inches thick, so that the circumference of the upper disc passes through the centre of the lower; determine, geometrically, the C. G. of the whole.

21. Why is it more dangerous to place luggage on the top of a coach than in the body?

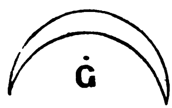


Fig. 55.

22. Explain how the properties of the centre of gravity are possessed by the point G of the figure (Fig. 55), which is not in the body itself.

23. Show that if a triangle be immersed in water so that the angular points are at distances below the surface equal to 8, 7 and 6 inches respectively, the C. G. is at a depth of 7 inches. (Apply first § 58 and then § 42.)

24. If a triangle be so situated that the distances of its angular points from a fixed plane are respectively h, k, and l, show that the distance of its C.G. from the same plane is $\frac{1}{3}(h + k + l)$.

25. If the crescent in the above figure (Fig. 55) be contained by two arcs, one of which is a semi-circumference of radius a, and the other the arc of a quadrant, prove that the distance of G from the line joining the points of the crescent

$$= a\left(\frac{\pi}{2} - 1\right),$$

having given that the distance of the C.G. of any segment from the centre of the circle of which it is a segment is proportional to the cube of the base divided by the area of the segment.

IX.—THE MECHANICAL POWERS.

Machines.

70. An instrument for transmitting a force from one point to another, or for changing the direction of motion, is a machine.

Any machine, however complex, is composed of a number of parts termed the simple mechanical powers.

Let us consider, for example, a flour mill. (Fig. 56.) On the left of the figure is the water-wheel, to which motion is first communicated. The moving force of the water is transmitted by the axle A to the first vertical toothed wheel, and by it to the horizontal toothed wheel attached to the axle B, which also carries a large toothed wheel C. This wheel communicates with a smaller one on the same shaft as the mill-stone, capable of being moved up and down upon the shaft. Thus there are three distinct parts to the machine; (1) the axle A, with its two wheels, the water-wheel to which the moving power is applied, and the toothed wheel to which is applied the resistance to be overcome; (2) the axle B, to the lower wheel of which the power is applied, and to the upper the resistance; (3) the axle N with a power-wheel moved by the wheel C, and the millstone to which is applied the final resistance. Now from the wheel which receives the force of the water to the stone which grinds the corn, the parts of the machine are so connected that the uniform movement of one part secures a like movement of the whole, and the equilibrium of the whole implies the equilibrium of each part. To trace *the conditions* of equilibrium of such a machine

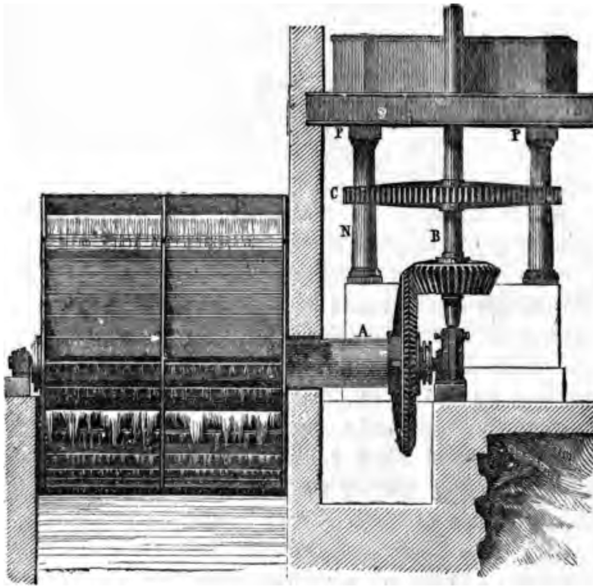


Fig. 56.

it is only necessary therefore to study the equilibrium of the simple machines.

71. It will be expedient to omit in this investigation several things which must be considered before a practical application of the principles can be made. For example, we shall first take no account of the friction between the different parts of the machine. Except where the contrary is stated, we shall neglect the weight of the machine itself. Cords and chains will be treated as if they had neither stiffness, thickness, nor weight. The consideration of these resistances is for the present deferred.

72. *The force* applied is technically called the

power, and the resistance, whatever may be its nature, is termed the *load*, or *weight*.

The condition of equilibrium in each machine will be expressed by stating the ratio of the power to the weight.

The Lever.

73. Any rigid rod movable about a fixed point is termed a *lever*. That there may be equilibrium with such a machine the forces acting on it must admit of a resultant passing through a fixed point in the lever. This point is termed the *fulcrum*, and the resultant of the force is the pressure sustained by the fulcrum.

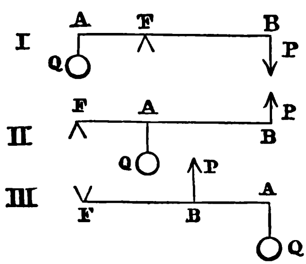


Fig. 57.

Levers are usually classified according to the position of the fixed point. If the fulcrum be between the power and the weight, the lever is of the *first* kind. If the weight be between the fulcrum and the power, the lever is of the *second* kind. If the power be between the fulcrum and the weight, the lever is of the *third* kind.

The crowbar of the mason, used as in Fig. 58, is a lever of the first kind.

The loaded barrow raised, as in Fig. 59, is a lever of the second kind.

The treadle of a sewing-machine is a lever of the third kind.

First, *let us suppose* the power and weight to be *parallel forces.*

Let P and Q be respectively the power and the weight applied at the points A and B. Let O be the fulcrum, and let A O $= a$, B O $= b$. Then, by the principle of

Fig. 58.

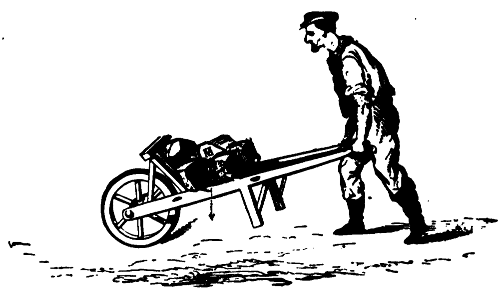


Fig. 59.

parallel forces, whichever kind of lever we consider, the equation of equilibrium will be

$$P \cdot a = Q \cdot b.$$

The pressure on the fulcrum in the lever of the first kind (Fig. 57 I.) is $P + Q$; in the lever of the second kind (Fig. 57 II.), $Q - P$; in that of the third kind (Fig. 57 III.), $P - Q$.

In all cases the greater force is attached to the shorter arm, so that Q is greater or less than P, according as a is greater or less than b. Hence, in lever 1, P may be greater or less than Q; in lever 2, since a is the whole rod P is necessarily less than Q; and in lever 3, P is necessarily greater than Q. If the forces be not parallel (Fig. 60) still the moments

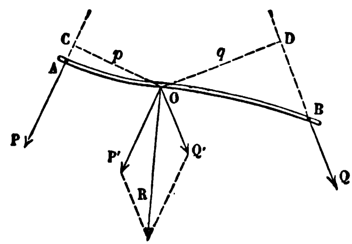


Fig. 60.

about the fulcrum in opposite directions must be equal, hence $P \cdot p = Q \cdot q$. To find the magnitude and direction of the pressure on the fulcrum O, suppose the forces P′ and Q′ at O equal and parallel to P and Q, then R, the resultant of these forces, will be the pressure on the point O.

Balances.

74. The balance is a lever of the first kind, having equal arms. The ordinary balance consists of a metallic bar termed the *beam* (Fig. 61), supported on a stand (c) by a sharp edge, called a knife edge, fixed horizontally in the middle of the beam. Attached to the beam at right angles is a needle or index, which moves

over a graduated arc, and is vertical when the beam is horizontal. Two dishes or scales (*b*, *b*) for receiving

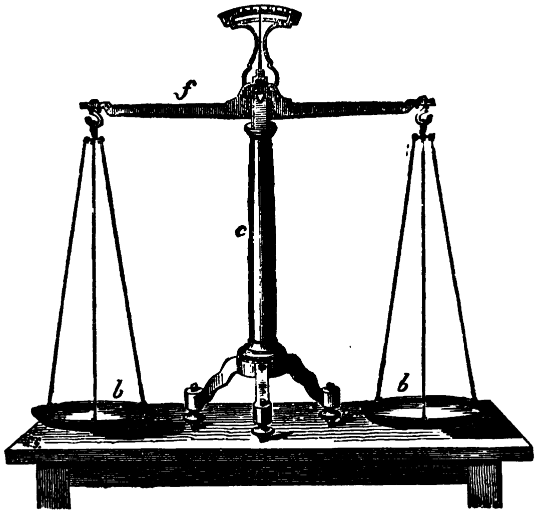


Fig. 61.

the weights are suspended from the extremities of the beam, also upon knife edges. These edges form the points of application of the weights. The distances of these points from the central edge are the arms of the balance. When equal weights placed in the scales always cause the beam to remain horizontal the balance is *true*. When this is not the case the balance is *false*.

To investigate the requisites of a good balance.

75. 1. When the scales are empty, the beam should *be horizontal* and index vertical. If the balance be

disturbed, the index should return again to the same place when the motion of the beam has subsided; hence the equilibrium should be stable. To secure this *the C. G. of the beam and its appendages should fall a little below the knife edge.*

2. When equal weights are placed in the scales the index should be vertical; hence *the arms must be of the same length.*

3. When the weights are unequal, the balance should easily indicate the inequality. The facility with which the index is turned from the vertical in such a case is termed the *sensibility* of the balance. The sensibility depends on two things. (1) The longer the arms, the greater will be the moment due to the excess of weight; hence *the arms should be as long as possible.* (2) When the beam is displaced, its weight tends to bring it back to the first position, and the greater the weight of the beam the less the amount of displacement caused by a given inequality in the weights; hence *the weight of the beam should be as small as possible.*

4. The moments of equal weights about the point of support should be the same, not only when the beam is at rest horizontally, but also when it is disturbed; hence the points of suspension of the scales should be in the same straight line as the point of support.

76. A form of balance now very commonly used is that indicated in Fig. 62. The dishes in this case are placed above the beam, so that the articles to be weighed may be more easily placed in them. The equality of the arms is here as indispensable as in the common balance. To prevent the scales from overturning, they are supported by rods, which descend

into the stand, and are attached by joints at their extremities to a second beam equal in length to the

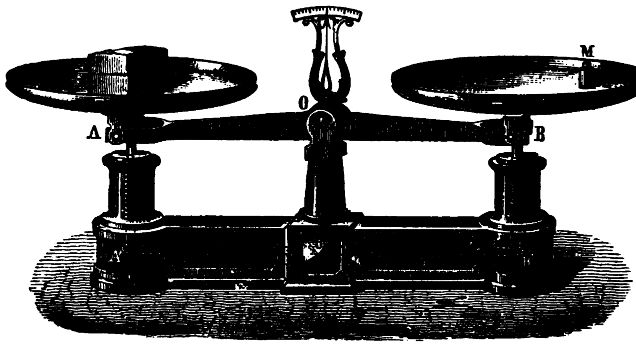


Fig. 62.

first. Though this balance is more convenient, it is less exact than the ordinary balance.

77. To test a balance, all that is necessary is to take weights apparently equal, and to transfer them. If they be not really equal, but appear so only in consequence of the inaccuracy of the balance, the fact will be made apparent when each is moved to the opposite scale.

The exact weight of a body may however be found by the aid of a false balance. Place the body to be weighed in one scale, and as much of some substance (sand for example) in the other as will make the beam horizontal. Now take out the body, replacing it by such weights of known value as will balance the sand. The sum of these weights will be the weights of the body. *We may, however,* find the true weight of a body by

weighing it in both scales. Let a and b be the arms of the balance, W the true weight of the body, x the

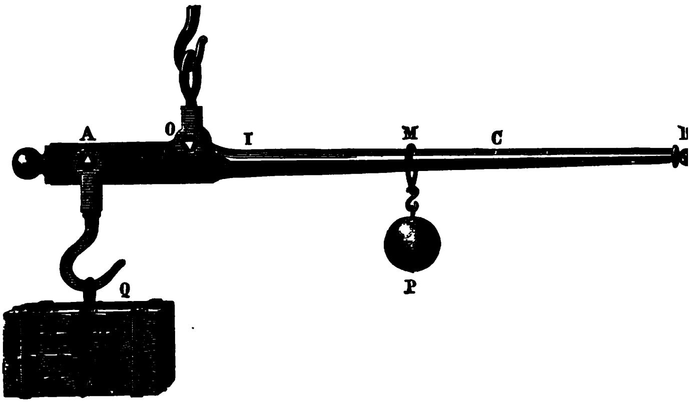


Fig. 63.

weight at b which balances W at a, and y the weight at a which balances W at b. Then, $W \cdot a = x \cdot b$

and $W \cdot b = y \cdot a$

On multiplying these equations we find that

$$W = \sqrt{xy}.$$

78. The common steelyard is a balance with unequal arms (Fig. 63). On one side of the knife-edge O, which is the fulcrum of the lever, the weight P remains the same, but its distance OM varies; on the other side, the weight suspended from the edge A may be changed, but the distance AO is constant. The condition of equilibrium, if we suppose the C. G. of the machine to be at O, will be $Q \cdot AO = P \cdot OM$. As Q varies on the one side, OM *may be* made to vary on the other, so that

the same weight P may balance various weights suspended from A. For example, let P in one position M balance 1 lb.; then, in a position M′ such that O M′ = 2·O M, P will balance 2 lbs. If the C. G. of the bar and its appendages be not at O, but at G, for example, then the moment of the weight of the bar W will be in the same direction as that of Q, and the condition of equilibrium will be $Q \cdot AO + W \cdot GO = P \cdot MO$.

Weighing Machines.

79. When the weights are considerable, machines composed of several levers are employed. Such is the weighing machine indicated in Fig. 64. It is com-

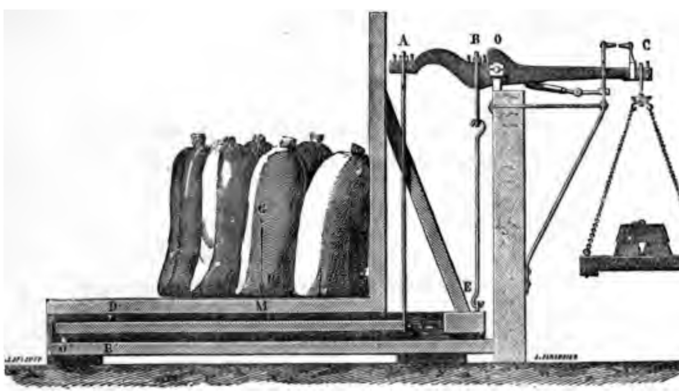


Fig. 64.

posed of three levers; A O movable about the point O, A′O′ movable about O′, and E D movable about D. The *lever E D bears* the platform upon which the object to

be weighed is placed. The point D is the extremity of a rod fixed to the lever O A′, the lever A C is bound to the others by the vertical rods A A′ and B E, the extremities being hinge-joints. A scale at C contains the weights.

Let P be the weight of the body, and p that of the weight in the scale C. Let P_1 be the pressure on D and

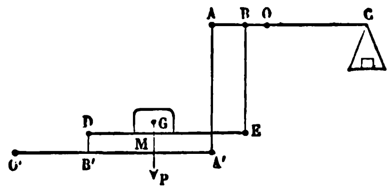


Fig. 65.

B′, P_2 the pressure on A and A′, P_3 the pressure on B and E.

I. From the equilibrium of D E we have $P \cdot DM = P_3 \cdot DE$.

II. From the equilibrium of A′ O′ we have $P_1 \cdot O'B' = P_2 \cdot O'A'$.

III. From the equilibrium of A C we have $P_2 \cdot AO + P_3 \cdot OB = p \cdot OC$.

Also $P_1 = P - P_3$, hence from I. we have

$$P_3 = \frac{P \cdot DM}{DE}, \text{ from II. } P_2 = P_1 \frac{\cdot O'B'}{O'A} =$$

$$(P - P_3) . \frac{O'B'}{O'A'} = P \cdot \frac{ME}{DE} \cdot \frac{O'B'}{O'A}$$

Hence, substituting in III.,

$$p \cdot OC = P \cdot \frac{ME}{DE} \cdot \frac{O'B'}{O'A'} \cdot AO + P \cdot \frac{DM}{DE} \cdot OB \cdot$$

$$= P \cdot OB \cdot \left\{ \frac{ME}{DE} \cdot \frac{O'B'}{O'A'} \cdot \frac{AO}{OB} + \frac{DM}{DE} \right\}$$

Now the balance is so constructed that

$$\frac{OA}{OB} = \frac{O'A'}{O'B'} \text{ or } \frac{OA}{OB} \cdot \frac{O'B'}{O'A'} = 1$$

$$\text{hence } \frac{ME}{DE} \cdot \frac{O'B'}{O'A'} \cdot \frac{AO}{OB} + \frac{DM}{DE} =$$

$$\frac{ME}{DE} + \frac{DM}{DE} = \frac{DE}{DE} = 1$$

$$\text{and } p \cdot OC = P \cdot OB.$$

It is evident, then, that if the arms OC and OB be taken so that $OC = 10 \cdot OB$, then P is always $10 \cdot p$, or if $OC = 112 \cdot OB$, P is as many cwts. as p is lbs., and similarly any other proportion may be secured.

The Wheel and Axle.

80. The wheel and axle is only a modification of the lever. This machine consists of two cylinders, A B and C D, of different radius (Fig. 66), having a common

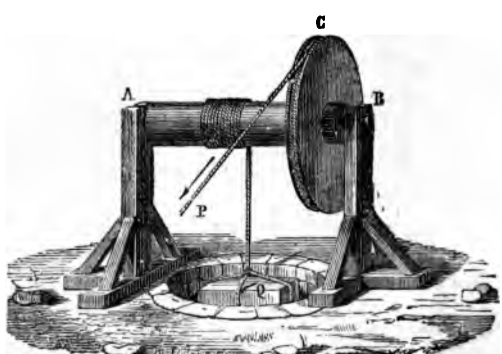


Fig. 66.

axis; the smaller being termed the axle, and the larger the wheel. A cord is wound round the wheel in one direction, and another cord round the axle in the opposite direction. To the former is attached the weight Q, and to the latter the power P. Suppose P and Q to be both vertical and represent the instrument as seen in the direction of the axis (Fig. 67). We have two parallel forces, P and Q, acting at the extremities of two arms, OB, OA, of a lever. The condition of equilibrium is therefore $P \cdot OB = Q \cdot OA$, or, if R and r be radii of wheel and axle, $P \cdot R = Q \cdot r$.

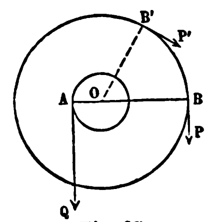


Fig. 67.

If the power does not act vertically but in the direction of the tangent at another point, as P′ (Fig. 67), then $P' \cdot R$ is the moment of P′ about O, and since $P'R = PR$, in this case also

$$P' \cdot R = Q \cdot r.$$

81. Instead of a wheel, one or several levers, l, l, (Fig. 68), may be placed in the axle as in the *windlass*.

82. When the axle is vertical, and the levers horizontal, the machine is termed a *capstan* (Fig. 69).

Toothed Wheels.

83. Several wheels and axles are frequently combined by means of spur wheels (Fig. 70). Several conditions must be satisfied in each pair of wheels. The *teeth on each* wheel must be equal to one another,

and equally distant, and the teeth of one wheel should be of the same size, and as far apart as those of the

Fig. 68.

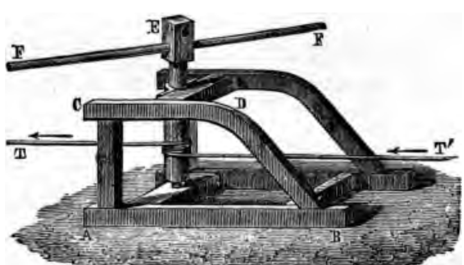


Fig. 69.

wheel in contact with it. The number of teeth will therefore be proportional to the circumferences or the radii of the wheels.

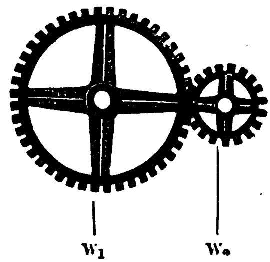


Fig. 70.

Equation of equilibrium for two-toothed wheels.

Suppose two such wheels to be in equilibrium with a weight W at the axle of the larger, and W_2 at the axle of the smaller. Let the radii of the wheels be R_1 and R_2, and let the radii of both axles be r. Also let P be the reaction between the wheels at their point of contact.

The equilibrium of the larger wheel gives $W_1 \cdot r = P \cdot R_1$

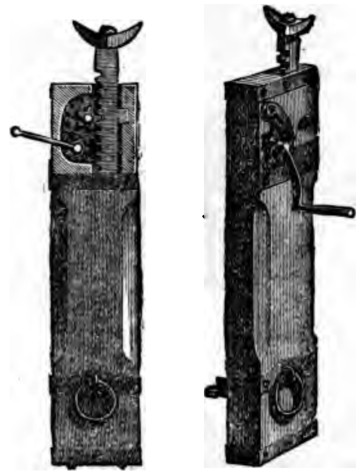
The equilibrium of the smaller . . . $W_2 \cdot r = P \cdot R_2$

therefore $\dfrac{W_1}{W_2} = \dfrac{R_1}{R_2} = \dfrac{\text{No. of teeth in larger.}}{\text{No. of teeth in smaller.}}$

84. Sometimes the teeth are placed on a straight bar instead of a wheel. Such a toothed bar is termed a *rack*, and the wheel in contact with it is called a *pinion*.

The *jack* (Fig. 71), used for lifting great weights through a small height, is a rack and pinion. The handle by which the rack is raised is termed a winch.

The instrument is an example of a double wheel and



Figs. 71 and 72.

axle; the winch takes the place of the first wheel, the lower pinion being its axle. The upper pinion is the second axle and supports the rack. The large toothed wheel of the upper axle works in gear with the lower pinion. The two pinions are usually, as in the figure, of the same size.

To find the equation of equilibrium.

Let P be the power applied to the winch, W the weight of the rack and its load, a the radius of the winch, b the radius of the large toothed wheel, r the radius of each pinion, and R the mutual reaction between the first pinion and the large wheel; then from the equilibrium of the first axle we have $P \cdot a = R \cdot r$, and from the equilibrium of the second axle we have $W \cdot b = R \cdot r$. *Therefore* $P \cdot a = W \cdot b$.

Exercises.

1. A weight of 30 lbs. balances a weight of 20 lbs. at the extremities of a straight lever 15 feet long; find the length of the arms.—*Ans.* 6 feet, 9 feet.

2. The arms of a lever are 7 in. and 9 in. in length, and the weight 3 lbs. is attached to the shorter arm; find the power.—*Ans.* $2\frac{1}{3}$ lbs.

3. If one end of a bar rest on a beam and a weight of 50 lbs. be suspended from it at one-fifth of its length from the beam, what power at the other end will support the weight, and what will be the pressure on the beam?—*Ans.* 10 lbs., 40 lbs.

4. A beam $4\frac{1}{2}$ ft. long is supported horizontally by two props at its extremities and produces a pressure of 4 lbs. on each prop; where must a weight of 36 lbs. be placed that the whole pressure on one prop shall be 10 lbs.?—*Ans.* 9 inches from one prop.

5. When two weights, 15 lbs. and 5 lbs., are suspended at the ends of a lever, the fulcrum is 9 feet from the smaller weight; where must the fulcrum be when the weights are each increased 5 lbs.?—*Ans.* 4 ft. from one end.

6. The C. G. of a wheelbarrow and its load, which weigh 100 lbs., is in a vertical line 18 inches from the centre of the wheel; what power applied to the handles at a distance of 3 ft. 6 in. from the centre of gravity will just lift the barrow?—*Ans.* 30 lbs.

7. A beam 12 ft. long balances about its middle point; about what point will it balance if a weight equal to twice that of the beam be placed at one end?

8. A beam A B 10 ft. long and weighing 56 lbs. balances about a point 3 ft. from A. When a weight is placed at B, the beam balances about a point 1·4 feet from B; find the weight.—*Ans.* 224 lbs.

9. The pressure on the fulcrum is 7, and the sum of the forces 13; find their distance from the fulcrum when the forces are 14 inches apart.—*Ans.* 6 in. and 20 in.

10. Find the true weight of a substance which, when placed in

one scale of a balance, seems to weigh 140 grammes, and in the other appears to weigh 154·35 grammes.—*Ans.* 147 grammes.

11. If in a balance one arm be ·98 of the other, and a body placed in the scale of the shorter arm balance 14·7 kilogrammes in the other scale, find the true weight of the body.—*Ans.* 15 kilogrammes.

12. The beam of a false balance is attached to one arm of a true balance, and weighs 2 lbs. A body placed in one scale of the false balance requires a weight of 3 lbs. in the other, and then the whole weighs 7·9 lbs.; find the weight of the body and the ratio of the arms.—*Ans.* 2·9 lbs.; 30 : 29.

13. The weight of a steelyard is 1 lb., the movable weight also 1 lb., the point of suspension of the body 8 inches, and the C. G. of the beam 3 in. from the fulcrum; graduate the beam for weights from 1 to 12 lbs.

14. What effect is made on the graduations by increasing the movable weight?

15. With a wheel and axle a power of 8 lbs. sustains a weight of 12 lbs.; what is the radius of the axle, that of the wheel being 24 in.?—*Ans.* 16 in.

16. The circumferences of wheel and axle are respectively 1 yard and 15 inches; what power will sustain a weight of 1½ tons?—*Ans.* 12·5 cwts.

17. Find the pressure on the catch of a ratchet-wheel (Fig. 72) 12 inches in diameter when it is attached to an axle 5 inches in diameter, sustaining a weight of 60 lbs.—*Ans.* 25 lbs.

18. A capstan turned by two horses is used to draw in a boat; the levers to which the horses are attached are 12 feet long, and the radius of the axle is 18 inches. When each horse is pulling with a force of 7½ cwts., find the tension of the cord attached to the boat.—*Ans.* 6 tons.

19. A uniform bent lever, the weights of whose arms are 5 lbs. and 10 lbs., rests with the shorter arm horizontal, what weight must be attached to the end of the short arm that the lever may rest with the long arm horizontal?—*Ans.* 37½ lbs.

The Pulley.

85. A pulley is a circular disc of metal or wood, capable of turning round an axis passing through its centre. Usually a groove is cut in the disc to keep a cord which passes over the pulley from slipping off (Fig. 73). Sometimes, however, the cord is replaced by a strap, and then the edge is convex (Fig. 74).

Fig. 73.

The pulley may be considered as a lever with equal arms, A O, O B (Fig. 75), so that forces P and P′ in equilibrium, at the extremities of the cord, are equal.

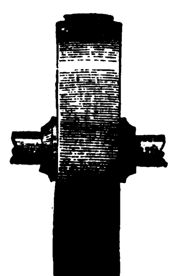


Fig. 74.

Whether the two parts of the cord be parallel or not these forces are equal, so that if one end of the cord be attached to a dynameter (Fig. 76), the indication of

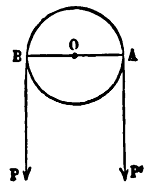


Fig. 75.

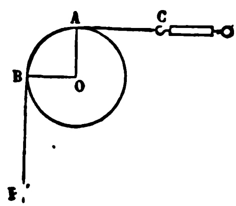


Fig. 76.

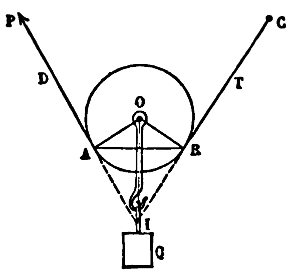


Fig. 77.

the instrument will be the same as if a weight equal to P were attached to it directly. A pulley, the axis of which is fixed in space (Fig. 73), is termed a *fixed* pulley, and in mechanics serves the purpose only of changing the direction of the power. If the axis be movable, the pulley is termed a *movable* pulley.

Let such a pulley with a weight Q attached, be supported by a cord C B A D (Fig. 77), fixed at a point C, and having at the other extremity a force P. The cord B C exerts a certain pressure T on the fixed point C, and this point reacts with a force equal and opposite. The pulley is then supported by the three forces P, T, and Q, which may be supposed to act at the points A B and O. That there may be equilibrium it is necessary that the directions of these forces shall meet in one point (§ 29). Let I be the point in the direction of Q, in which P and T meet.

The triangles O A I and O B I are equal in every respect, and therefore O I bisects the angle A I B. It follows that the forces P and T are equal, and, consequently, if we know the angle I, and the weight Q, we can find P, or if we know P and the angle I, we can find Q. When the cords A'D, B C are parallel, Q is the resultant of two equal parallel forces, and therefore $2\ P = Q$. In this case the pressure on the fixed point C is also equal to $\frac{1}{2}$ Q.

Several movable Pulleys with separate strings.

86. In this system of pulleys each movable pulley has a string of its own, one end of which is attached to the beam, and the other end to the next pulley, the power being applied to the free end of the cord passing over the highest pulley (Fig. 78). If we neglect the weights of the pulleys, and take T, T', T'' to represent the tension of each separate cord, then

$$T = \frac{1}{2}Q\,;\ T' = \frac{1}{2}T = \frac{1}{4}Q\,;\ T'' = \frac{1}{2}T' = \frac{1}{8}Q$$

$$\text{and } T'' = P\,;\ \text{hence } P = \frac{1}{8}Q.$$

If there be n movable pulleys, then $P = \frac{1}{2^n}\cdot Q$.

If we are required to take into account the weights W W' W'' of the pulleys, then

$$T = \frac{1}{2}(W + Q);T' = \frac{1}{2}(W' + T);P = \frac{1}{2}(W'' + T')$$

therefore if there be n pulleys

$$P = \frac{1}{2^n}\ Q + \frac{1}{2^n}W + \frac{1}{2^{n-1}}W' + \frac{1}{2^{n-2}}W'', \text{\&c.}$$

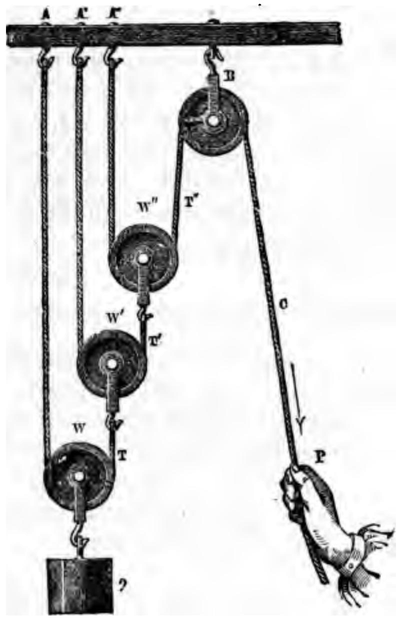


Fig. 78.

Several movable Pulleys in one block, the same string passing round all.

87. In this case we suppose the parts of the string between the pulleys to be parallel. The tension of the string is the same throughout, and is equal to P; hence, if, as in the figure (Fig. 79), there are three movable *pulleys, six parts* of the string support the weight,

Fig. 79.

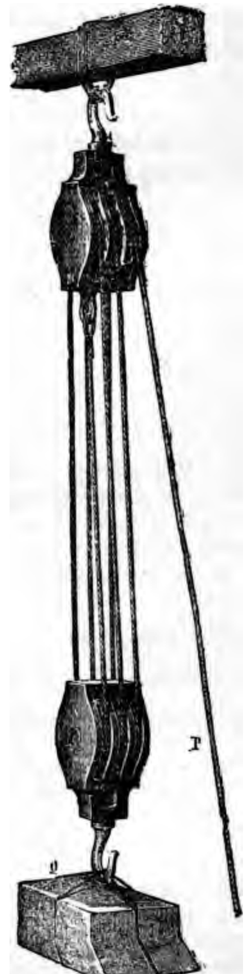


Fig. 80.

each bearing $\frac{1}{6}$; hence $P = \frac{1}{6} Q$. If there be n strings from the lower block

$$P = \frac{1}{n} Q.$$

The weight of the pulley will be taken into account by adding it to Q.

Exercises.

1. A movable pulley supports a weight of $5\sqrt{3}$ lbs. and the two parts of the cord make an angle of 60°; find the power.

Three forces act at the point I (Fig. 77), it will be convenient to take moments about a point in the direction of one of them.

Let B be the point chosen, then the moment of T about B is O.

The perpendicular from B on O I as $\frac{1}{2}$ A B.

Since $\angle$ A B I = 60° the triangle A B I is equilateral, and the perpendicular from B on A I = $\frac{\sqrt{3}}{2}$ A B.

$$\text{Hence } P \times \frac{\sqrt{3}}{2} AB = Q \times \frac{1}{2} AB$$

$$P \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \times \frac{1}{2}$$

therefore P = 5.

2. Find the power which will support a weight of 611 lbs. with three movable pulleys, having separate parallel cords *attached* by one end to the beam, the pulleys taken in order from *the lowest weighing* respectively 7, 5, and 3 lbs.

```
  611
    7
2)618
  309
    5
2)314
  157
    3
2)160
   80
```

The formula of § 86 shows that the power will be found by adding the weight of the lowest block to the load Q, and dividing by 2; adding the weight of the next pulley, and again dividing by 2; and so on for all the blocks.

3. In the last example if the weights of the pulleys are 8, 6, and 4, and P = 34 lbs., find the load.

```
 34 × 2
 68
  4
 64 × 2
128
  6
122 × 2
244
  8
236
```

Here the process is reversed, instead of dividing by 2, we multiply; instead of adding the weights of the pulleys we subtract them, commencing with the highest.

4. What power will sustain 1 ton with a block and tackle, the block containing 3 sheaves (pulleys)?—*Ans.* $\frac{1}{6}$ ton.

5. What is the weight of a block containing 4 sheaves, by means of which 20 lbs. will balance 140 lbs.?—*Ans.* 20 lbs.

6. If 4 movable pulleys are used with the arrangement represented on Fig. 78, and the weights taken in order from the lowest, are 7, 3, 8, and 5 lbs., find the power which will sustain a weight of 395 lbs.—*Ans.* 30 lbs.

7. In the above case, find the pressure on the beam.—*Ans.* 448 lbs.

8. Which would be the most advantageous arrangement of the 4 pulleys?

9. A man weighing 154 lbs. sustains a weight of 333 lbs. with 3 *movable pulleys*, each having a separate string attached at

one end to the beam. The pulleys weigh respectively 3 lbs., 2 lbs., and 1 lb.; find the pressure exerted by the man on the floor on which he stands.—*Ans.* 111 lbs.

10. If the man in the above example exerts a pressure on the floor of 100 lbs., what weight does he sustain?—*Ans.* 421 lbs.

11. Find the power which will sustain a weight of 90 lbs. with a single movable pulley C, the cord making an angle of 60°.—*Ans.* $30\sqrt{3}$.

12. Two pulleys A and B fixed to a beam at a distance equal to the diameter of either of them, are used with a movable pulley C of twice this diameter to support a weight Q. The cord is fixed by one end to C, passes round A, then round C, then round B; find the power.—*Ans.* $\frac{1}{3}$ Q.

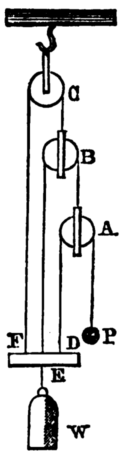


Fig. 81.

13. With a system of pulleys in which a separate string passes round each pulley, and is attached by one end to the weight, show that if we neglect the weights of the blocks the tensions in the different strings are

$$P, 2P, 4P, 8P, \&c.$$

$$\text{and that } W = P\,(1 + 2 + 2^2 + \&c.)$$

14. In the above system if there be n pulleys

$$W = P\,(2^n - 1).$$

15. If W = 147 lbs., A = $1\frac{1}{2}$ lbs., and B = $2\frac{1}{2}$ lbs., find P.—*Ans.* 20 lbs.

16. If in this system there be 5 pulleys, and the weights of the 4 movable blocks commencing with the lowest be 2, 3, 4, and 5 lbs., find the weight which will be sustained by a power of 12 lbs.—*Ans.* 440 lbs.

17. Show that this arrangement possesses the advantage over that represented in Fig. 78, that in the former case the weights of the blocks assist the power, and in the latter oppose it.

The Inclined Plane.

88. If a body be pressed against a hard smooth surface, the resistance offered by the surface will be at right angles to it. When therefore a body is placed on a smooth horizontal surface it may be supported, for the weight and the reaction may be in the same straight line, and if they be equal and opposite will then be in equilibrium. When, however, the plane is inclined, a third force will be necessary to produce equilibrium.

Let M be a body of weight W, supported on a smooth inclined plane A B, by a force F, the direction of which meets the vertical through the C. G. of the body in the point O. That there may be equilibrium, the forces must admit of a resultant R along the normal to the surface, hence they must be in the same plane as the normal. Take a line O D to represent the weight, and through D draw D E parallel to the force F and meeting the normal in E, then the sides of the triangle O D E represent the forces.

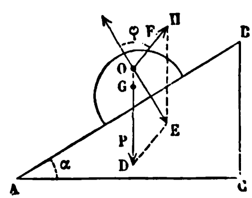


Fig. 82.

$$\text{Therefore } \frac{F}{W} = \frac{DE}{DO} \text{ and } \frac{R}{W} = \frac{EO}{DO}$$

The sides of the angle D O E are respectively perpendicular to the sides of the angle A, and, consequently, these angles are equal. The angle O E D is equal to the angle *between the* force F and the normal; hence the

triangle O D E is determined when we know the weight, the inclination of the plane, and the direction of the force. There are two cases in which the solution of the triangle is very simple.

1st. When the force F *acts in a direction parallel to the plane.*

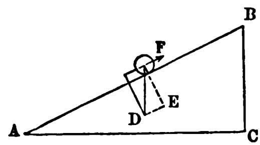


Fig. 83.

In this case the angle D E O is a right angle, and the triangle O D E is similar to the triangle A B C, the side D E corresponding to B C, and D O to A B.

Let $h =$ B C, the height of the plane;
$l =$ A B, the length of the plane;
$b =$ A C, the base of the plane;

therefore $$\frac{F}{W} = \frac{DE}{DO} = \frac{BC}{AB} = \frac{h}{l}.$$

2nd. When the force F *is horizontal.*

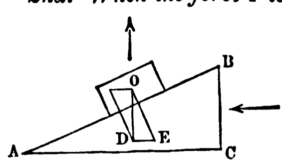


Fig. 84.

In this case the angle O D E is a right angle, and O E D is similar to A B C, D E corresponding to B C and O D to A C.

Therefore $$\frac{F}{W} = \frac{DE}{DO} = \frac{BC}{AC} = \frac{h}{b}.$$

In all these cases the plane is supposed to be fixed. In the second case, instead of the plane being fixed and the body movable, the plane may be movable horizontally and the body vertically. In this case *the horizontal* force must be applied *to the plane*

(Fig. 84) instead of to the body. The condition of equilibrium remains the same, namely, that

$$\frac{F}{W} = \frac{\text{height}}{\text{base.}}$$

The Wedge.

89. A wedge is a triangular prism used as a movable inclined plane, and is employed to separate bodies that are urged together by great pressures (Fig. 85). The resistances to be overcome act perpendicularly to the faces in contact with the body, and the force is usually applied perpendicularly to the base; hence a section of the wedge perpendicular to the faces sustaining the pressure is a triangle, the sides of which are perpendicular to the forces, and therefore, when there is equilibrium, proportional to the forces. Let P be the pressure necessary to keep the wedge in its place, R the pressure on each face, b the breadth of the base of the triangular section, l the length of one of the equal sides; the equation of equilibrium is therefore

Fig. 85.

$$\frac{P}{R} = \frac{b}{l}$$

In practice, however, the friction between the surfaces plays such an important part, and is so large compared with the power, that the above proportion cannot be applied with any degree of accuracy. Again, the power usually employed with the wedge is not pressure but percussion, and we cannot accurately state the relation between the force of a blow and the *resistance it overcomes.* Nevertheless one fact shown to be true

when friction is neglected is also found to be true in practice, viz. that the more acute the angle of the wedge the more powerful is the instrument.

The Screw.

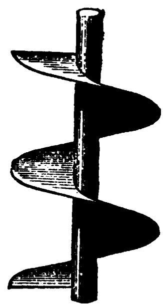


Fig. 86.

90. When an engineer wishes to make a road to the top of a very steep hill, he carries the road round the hill, making it ascend gradually. In a similar manner a path may be made from the bottom to the top of a column by a winding shelf or ledge. Such an arrangement would be a screw (Fig. 86).

A screw may be considered as an inclined plane wound round a cylinder.

The projecting coils are termed the *threads* of the screw. They may be square as in Fig. 88, or triangular as in Fig. 87.

The distance between the upper edge of one thread and the corresponding edge of the next, measured on a line parallel to the axis, is termed the *distance between the threads.* The screw is usually connected with a concave cylinder termed a nut, on the interior surface of which a spiral cavity is cut, corresponding exactly to the thread of the screw which moves in it.

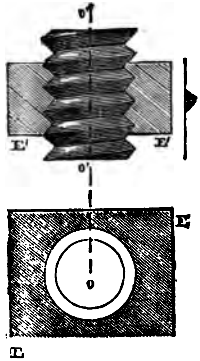


Fig. 87.

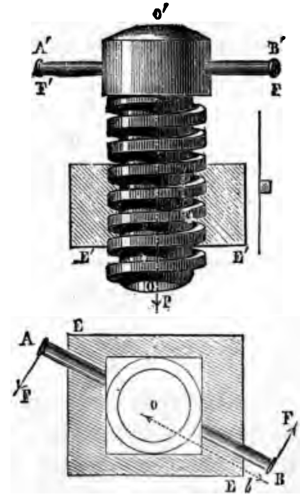


Fig. 88.

The Condition of Equilibrium when a weight is supported by a Screw.

91. Let the weight W be supported by a force P, applied at the circumference of a screw in a direction perpendicular to a plane through the axis. The arrangement is equivalent to an inclined plane acted on by a horizontal force (Fig. 84). The condition of equilibrium is in this case $\frac{P}{W} = \frac{\text{height}}{\text{base.}}$

Now suppose the screw to be unrolled from the cylinder. The whole inclined plane thus formed will be similar to that portion of it which would go exactly once round the cylinder. Let A B C be such a portion (Fig. 90).

Then AC is the circumference of the cylinder, BC the distance between the threads. Let AC = C, BC = d

$$\text{then } \frac{P}{W} = \frac{d}{c}$$

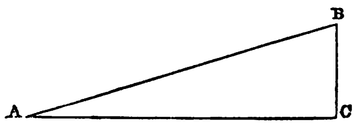


Fig. 89.

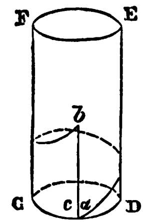


Fig. 90.

The screw is, however, rarely used without a lever. Let F be a power applied at the end of a lever, the length of which from the axis of the cylinder is l, and let r be the radius of the cylinder. Let P be a force applied at the circumference of the cylinder, which would have the same effect as F, then by the principle of the lever, $P \cdot r = F \cdot l$

$$\text{or } P = F \cdot \frac{l}{r}$$

Substitute for P in the above equation, and remark that $C = 2\pi r$, therefore

$$\frac{Fl}{Wr} = \frac{d}{c} = \frac{d}{2\pi r} \text{ hence } \frac{F}{W} = \frac{d}{2\pi l}$$

or the power is to the weight as the distance between the threads is to the circumference of the circle described by the power.

We may, therefore, increase the mechanical force of the screw, either by diminishing the distance between *the threads or by* lengthening the arm of the lever, or *by both.*

The screw is commonly used to exert pressure. The press represented in Fig. 91 is a familiar example. The

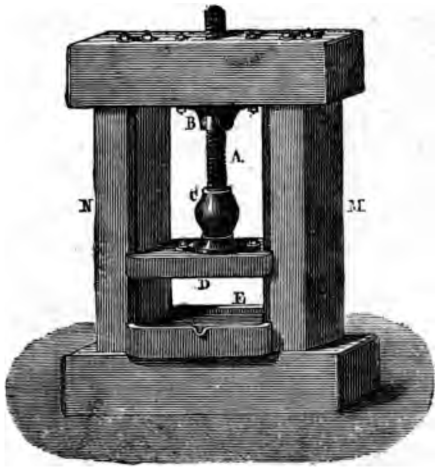


Fig. 91.

upper beam supports the nut which is fixed. The head C of the screw, moving in a socket fixed to the block D, is pierced for the insertion of levers.

The screw is sometimes associated with a spur wheel, and is then termed an *endless* screw, because the teeth of the wheel succeed and replace each other as they advance, so that they never arrive at the end of the screw.

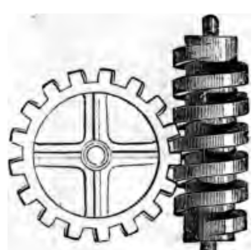


Fig. 92.

Compound Machines.

92. The simple machines admit of an endless variety of combinations.

Several levers may be combined, so that the force is transmitted from one to the other (Fig. 66); a combination of wheels and axles forms the *crab* in Fig. 94 and the mill of Fig. 55.

The crane used for lifting heavy weights consists of wheel and axle, and a movable pulley. A is the shaft

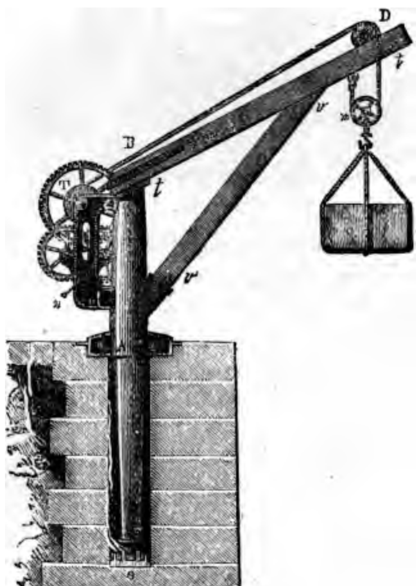


Fig. 93.

of the crane resting below on a pivot, and supported in the middle by a metal ring let into a block of stone. The oblique beams *t t*, *v v*, form the *arm* of the crane.

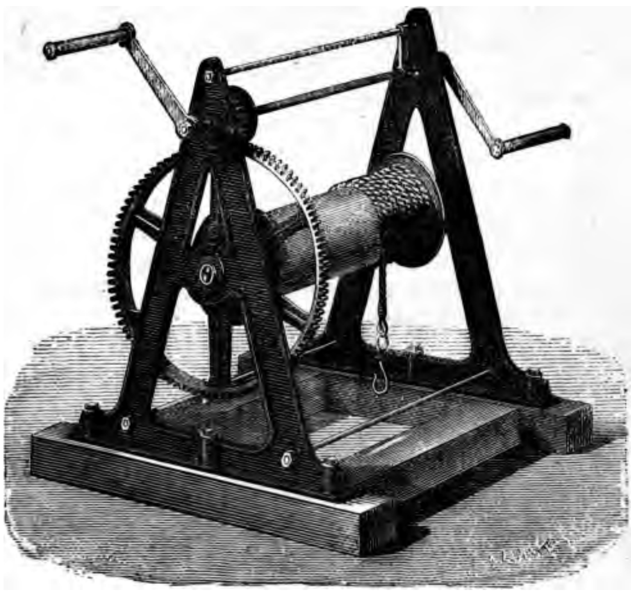


Fig. 94.

The wheel and axle may be combined with the inclined plane as in Fig. 95.

The condition of equilibrium in a compound machine is obtained by finding the equations of equilibrium of each part and eliminating the unknown reactions between them. *For example,*—Find the equation of equilibrium when

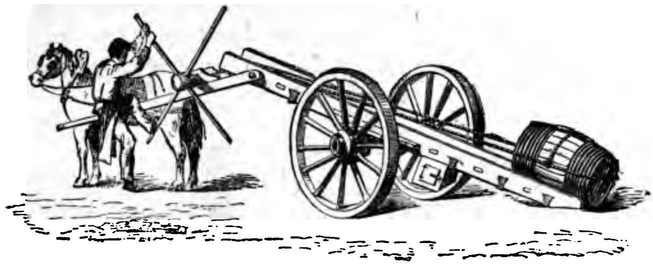


Fig. 95.

two weights W_1, W_2, are supported on a double inclined plane by a cord passing over a pulley at the common vertex of the planes. Let A B C, A′ B C be the planes, let T be the tension of the cord. From the equilibrium of W_1, it follows that

$$\frac{T}{W_1} = \frac{BC}{AB}$$

and from the equilibrium of W_2 it follows that

$$\frac{T}{W_2} = \frac{BC}{A'B}$$

dividing the second equation by the first

$$\frac{W_1}{W_2} = \frac{AB}{A'B}$$

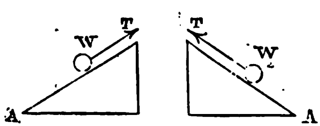


Fig. 96.

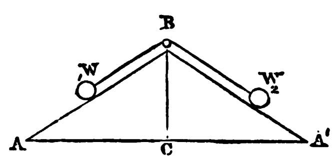


Fig. 97.

By the *mechanical advantage* of any machine is meant the ratio of the weight to the power, when in equilibrium; thus, if a power of 5 lbs. sustain a weight of 80 lbs. the mechanical advantage is 80 ÷ 5, or 16. Thus the mechanical advantage of a simple machine is found by inverting the preceding equations of equilibrium.

To find the mechanical advantage of any combination of machines.

In a combination of machines, the weight of the first machine is the power of the second, the weight of the second the power of the third, and so on. Let a, b, c, be the separate advantages of three machines in combination. Let P and Q be the power and weight in the first, Q and R those in the second, R and W those in the third; then, $Q = a\,P$, and $R = b\,Q$, and $W = c\,R$, multiplying together $Q\,R\,W = a\,b\,c\,P\,Q\,R$; $\therefore\ W = a\,b\,c\,P$. Therefore $a\,b\,c$ represents the mechanical advantage of the combination; that is, the advantage of the combination is equal to the product of the separate advantages of the component machines.

In like manner it may be shown that the mechanical advantage of the combination of any number of machines is equal to the product of the separate advantages of the component machines.

To find the mechanical advantage of the Crane. (Fig. 93.)

Let l be the length of the arm of the winch, and a the radius of its axle, then advantage of winch and axle $= \dfrac{l}{a}$

Let R_1 be the radius of the first, and r_1 the radius of its axle, then the advantage of first wheel and axle $= \dfrac{R_1}{r_1}$

Let R_2 be the radius of the second wheel, and r_2 the radius of its axle, then advantage of second wheel and axle $= \dfrac{R_2}{r_2}$

advantage of pulley $=2$.

Consequently, advantage of crane $= \dfrac{2\,l\,R_1 R_2}{a\,r_1\,r_2}$

If N_1, N_2, be the number of teeth respectively in the two wheels, and n_1, n_2, those in the axles of the winch and of the first wheel, then $\dfrac{R_1 \cdot R_2}{a\,r_1} = \dfrac{N_1\,N_2}{n_1\,n_2}$,

and therefore advantage of crane $= \dfrac{2\,l \cdot N_1\,N_2}{n_1 \cdot n_2 \cdot r_2}$.

Exercises.

1. If the power represented by P act in a direction parallel to the plane, and support a weight W, solve the following:—

   1. W = 100 lbs., height = 3 ft., length = 4 ft.; find P.—*Ans.* 75 lbs.
   2. W = 122 lbs., height = 11 in., base = 5 ft.; find R.—*Ans.* 120 lbs.
   3. P = 35 lbs., height = 15 in., length = 27 in.; find W.—*Ans.* 63 lbs.
   4. P = R; find the inclination of the plane.—*Ans.* 45°.
   5. P = 160 lbs., height = 32, base = 255; find W.—*Ans.* 1,285 lbs.
   6. W = 14,267 lbs., height = 72 ft., base = 1,295; find P.—*Ans.* 792 lbs.
   7. A railway train, weighing 50 tons, is supported on an inclined plane, rising 1 ft. for every 50 ft. of length, by a rope attached to a stationary engine; find the tension.—*Ans.* 1 ton.

2. Solve the following exercises, the direction of the power being horizontal:—

   1. W = 15, height = 6, base = 9; find P.—*Ans.* 10.
   2. W = 36, height = 5, length = 13; find P.—*Ans.* 15.
   3. P = 3·5, base = 24, length = 25; find W.—*Ans.* 12.
   4. W = 9, height = 28, base = 45; find R.—*Ans.* 10·6.
   5. P = $\frac{1}{2}$ R; find the inclination.—*Ans.* 30°.
   6. P acting parallel to the plane, or 2 P acting horizontally, will support W; find the inclination.—*Ans.* 60°.

3. A weight of 253 lbs. is supported on an inclined plane rising 88 ft. in a length of 137 ft. by two equal forces, one being horizontal and the other parallel to the plane. Find the forces.—*Ans.* 92 lbs.

4. A uniform beam, A B, rests with one end, B, on an inclined plane 12 ft. in height, to 37 ft. of length, and the other end, A, on a horizontal plane, and is kept from sliding by a horizontal force, F, applied at A. The beam measures 20·2 ft. in length, and weighs 617·5 lbs. The end, B, is 12 ft. 4 in. from the foot of the plane; find F.—*Ans.* 99 lbs.

NOTE.—Take moments about the point B, then if $R =$ the reaction of the surface at A

$$R \times 19{\cdot}8 = W \times 9{\cdot}9 + 4\,F.$$

Next resolve the forces in the direction of the plane

$$(W - R) \times 12 = F \times 35$$

then eliminate R between the two equations.

5. If the inclination of the plane be 30°, and the power be inclined at an angle of 30° to the plane, find the ratio of the power to the weight.—*Ans.* $\frac{1}{\sqrt{3}}$

6. A weight is supported on an inclined plane by two forces, each equal to half the weight, one acting along the plane, and the other horizontally; show that the height is to the length as 4 to 5.

7. The diameter of a screw is 7 in., and the distance between the threads one-fourth of an inch; what power applied at the circumference of the screw will support a weight of 110 lbs.?—*Ans.* 1·25 lbs.

8. When the circumference of the screw is 12 inches, and there are three threads to the inch, find the weight which will be supported by a power of 10 lbs.—*Ans.* 360 lbs.

9. In a common press the diameter of the screw is 6 in., the distance between the threads $\frac{3}{8}$ in., and the length of the lever, measured from the axis, is 4 ft.; what power will support a resistance of 352 lbs.?—*Ans.* ·7 lbs.

10. If the circumference described by the end of the lever be 10 ft., the power 10 lbs., and there be three threads in 2 in., find the resistance supported.—*Ans.* 1,800 lbs.

11. If a screw be formed upon a cylinder whose length is 10 in., and circumference 4 in., how many turns must be given to the thread, in order that the power may be one-eighth of the weight?—*Ans.* 20.

12. If the endless screw (Fig. 92) be moved by a handle which describes a circumference c, and the distance between the threads be d, then when a weight W, hanging from a cord round the axle of the spur wheel, is supported by a power P at the handle—

$$\frac{P}{W} = \frac{d \times \text{radius of axle.}}{c \times \text{radius of wheel.}}$$

X.—PASSIVE RESISTANCES.

Friction.

93. In the preceding investigations we have disregarded the resistances arising from the roughness of surfaces, the rigidity of cords, and the presence of the air or water. We proceed to consider the effects of these resistances.

No surfaces are perfectly smooth. When a body is laid on a horizontal surface, even though it be a surface of polished steel, some force is necessary to make the body slide. The resistance to be overcome in consequence of the roughness of the surfaces is termed *friction.*

In order to show how to measure friction, let us imagine a plane C D movable about a point C, and resting on a screw A B, by means of which it can be gradually inclined. When C D is horizontal, place upon it a body of weight W, and then turn the screw A B. We may give to the plane a certain inclination without causing the body to slide down it. If we take a line G P to represent the weight of the body, we can decompose the weight into two other forces represented by $P'G$, $P''G$ respectively, perpendicular and parallel to the plane. The first of these presses the body against the plane, and will be counteracted by the resistance of the plane; the second tends to make the body slide down the plane, and will be counteracted by the friction *between the two* surfaces.

Let F = the force of friction, R = the reaction,

$$\text{then } F = R \times \frac{AB}{BC} \text{ (§ 88.)}$$

When the plane is horizontal the pressure on the plane is the whole weight, and the part P P′ is O. As the plane is raised, P P′ increases and P′G decreases, hence the ratio $\frac{PP'}{P'G}$ increases. There is, however, a limit to this ratio, for the plane will reach such an in-

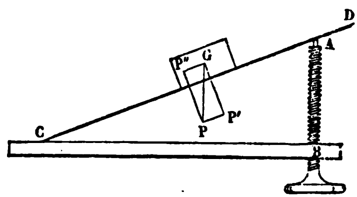


Fig. 98.

clination that the friction will not prevent the sliding of the body. The greatest angle at which the plane will support the body is termed the *angle of repose.* Suppose the plane to be inclined at that angle, and therefore the body to be on the point of sliding, then the ratio $\frac{PP'}{P'G}$ is termed the *coefficient of friction.* Let B C be taken a foot in length, then if we keep the same unit $\frac{PP'}{P'G} = \frac{AB}{BC} = \frac{AB}{1}$; hence the length of A B expressed as the fraction of a foot is the coefficient of friction. Now it is a law of friction that so long as the substances composing the surfaces in contact remain the same when the body is on the point of moving, the ratio of the friction to the pressure between the surfaces

is the same. It is therefore always equal to the coefficient of friction. It must be borne in mind that the coefficient of friction varies for different surfaces, and can only be determined by experiment in the way described above. Hence the first law of friction is as follows:—

1. *When the materials composing the surfaces in contact remain the same the friction varies as the pressure.*

Suppose, for example, that a block of wood having a hole bored in it, rests on a plane inclined at the angle of repose; if lead be poured into the hole, the screw may be turned so as to incline the plane at a greater angle without causing the body to slide. By increasing the pressure we increase the friction.

2. *The friction is independent of the extent of the surfaces in contact.*

For example, in the case supposed above, the angle of repose will be found to remain the same whichever face of the body is placed in contact with the plane. This result may at first appear surprising, but on reflection it will be seen that it is a natural consequence of the first law; for if, in the second position, the points in contact be doubled, each bears only half the pressure, so that the friction at each point in the second case is half of that in the first, and the total friction remains the same.

When the surfaces have been long in contact this rule does not hold. When pieces of timber are morticed together in building constructions, the parts acquire a force of adhesion and cohesion which is not *proportional to* the pressure.

3. When the body is in motion the friction is independent of the velocity.

The effect of friction is always to resist the motion of the body; hence, if the object of a force be to move a weight, friction opposes the power, but if it be applied to keep a body at rest, a less power will be sufficient than if there were no friction.

Although friction in a machine is a disadvantage, it is the source of the efficacy of such instruments as nails, pegs, wedges, &c., for when a wedge is driven into a substance by the force of percussion, it would rebound after each blow but for friction.

Friction is frequently utilised when great resistances are required to prevent motion. For example:—A boat carried in the current of a stream might be easily arrested by making two or three turns of the rope attached to it round a tree or fixed object.

When one surface slides on another the resistance is termed *sliding* friction; when one rolls on the other, so that different points in each are brought into contact, the friction is termed *rolling* friction. With the same surfaces and pressure sliding friction is much greater than rolling friction. On this account carriages are supplied with wheels, household furniture moves on

Fig. 99.

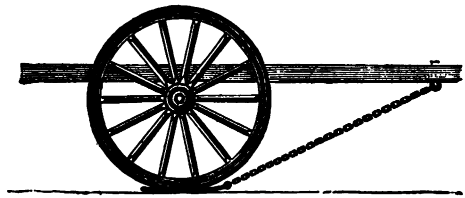


Fig. 100.

castors; the shaft of the crane in Fig. 93 is supported by friction rollers (Fig. 99).

When, as in descending a steep hill, it is advisable to check the motion of the carriage, the wheel is locked by a chain (Fig. 100), or by a break (Fig. 101), so that the friction may offer a greater resistance to the motion.

For the same end, railway carriages are supplied with breaks, consisting of blocks of wood made to come into close contact with the hoop of the wheel by the action of a lever.

The friction is a maximum when the wheels are completely locked.

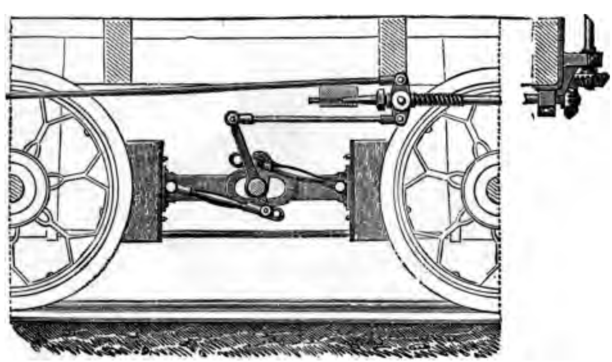


Fig. 101.

Example 1. A carriage weighs two tons, and the greatest force which may be applied to it horizontally without moving it is 15 cwts.; find the coefficient of friction. Here the pressure on the plane is equal to the weight, hence if F be the greatest force, which is balanced by friction, and e the coefficient $F = W\,e$, therefore $15 = 40\,e$ or $e = \frac{3}{8}$.

2. What force would draw the carriage up a plane rising 9 in 41? By § 88 the pressure on the plane $= W\,\dfrac{base}{height} = \dfrac{2 \times 40}{41}$, hence friction $= \dfrac{2 \times 40}{41} \times \frac{3}{8} = \dfrac{30}{41}$ tons, and the force must exceed this resistance.

3. What are the forces acting on a body which stands at rest on the side of a hill?

4. A ladder, A B, has one end, A, on a rough horizontal road, and the other end, B, against a rough vertical wall; what are the forces acting on it?

5. A body will just rest on a plane inclined at an angle of 45°; find the coefficient of friction.

6. If the coefficient of friction be $\dfrac{1}{\sqrt{3}}$, show that the inclination of the plane is 30°.

7. The height of a rough inclined plane is to the length as 3 to 5, and a weight of 30 lbs. is supported by friction only; find the force of friction.—*Ans.* 18 lbs.

8. A weight of 100 lbs. rests on a horizontal plane, and the coefficient of friction is ·2; find the horizontal force which will move the weight.

9. A ladder 45 ft. long, rests with one end on a rough horizontal plane, and the other against a smooth vertical wall. The C. G. of the ladder is 25 ft. from the foot, and the ladder is on the point of sliding when the foot is 27 ft. from the wall. Find the coefficient of friction between the ladder and plane. —*Ans.* $\frac{5}{12}$.

10. Show that when a body rests on a horizontal plane, the smallest pressure that will bring it into the state bordering on motion will act in a direction inclined to the horizon at an angle equal to the limiting angle of resistance.

XI.—VIRTUAL VELOCITIES.

94. When a force P supports a resistance Q by means of any machine, if by moving the power through a space S_1 the weight is brought through a space S_2, then $P \cdot S_1 = Q \cdot S_2$, or the spaces are inversely as the forces. This fact is frequently stated thus,—*what is gained in power is lost in speed.*

We will prove this remarkable proposition in the case of the simple machines.

The Lever. Let A B be a lever, of which C is the fulcrum (Fig. 102), and let a power P at A balance a

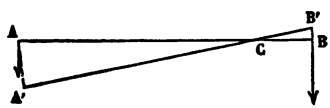


Fig. 102.

weight Q at B. Suppose the lever to move about the fulcrum, then A will describe an arc A A′, and B an arc B B′. Let A A′ = S_1 and B B′ = S_2. It has been proved that

$$\frac{P}{Q} = \frac{CB}{AC};$$

but since the angles at the common centre C are equal, the arcs S_1 and S_2 are proportional to their radii, hence

$$\frac{CB}{AC} = \frac{S_2}{S_1}$$

and therefore $P \cdot S = Q \cdot S_2$.

When the displacement is small, the arcs may be *supposed* to coincide with their chords, and S_1, S_2 may

be taken as the linear displacements of the points A and B.

Again, if s'_1 and s'_2 be the vertical displacements of the points A and B, they are respectively the heights of the similar triangles A C A′ and B C B′, and are proportional to the chords A A′, B B′; hence $P \cdot s'_1 = Q \cdot s'_2$.

A precisely similar demonstration would prove the principle for the wheel and axle.

Pulleys. Let us take first the case in which the same cord passes round all the pulleys and the parts of the cord are parallel (Fig. 80). Let there be six strings to the lower block. Suppose the weight raised throught a height s_2, then each of the six strings is shortened by s_2, and therefore the power must descend through $6 \cdot s_2$.

Hence it follows that $\dfrac{S_2}{S_1} = \dfrac{S_2}{6S_2} = \dfrac{1}{6}$

Now it has been proved that

$$\frac{P}{Q} = \frac{1}{6};$$

and therefore $P \cdot S_1 = Q \cdot S_2$.
In the same way the principle may be proved, whatever the number of strings.

Consider now the arrangement in which each pulley hangs by a separate string (Fig. 78). Let there be three movable pulleys. Suppose the weight raised through a height s_2. The lowest pulley is raised through the same height, the next pulley $2\,s_2$, the third pulley $4\,s_2$, and therefore the point at which the power acts is raised through $8\,s_2$.

$$\text{Hence } \frac{S_2}{S_1} = \frac{S_2}{8\,S} = \frac{1}{8}$$

but it has been proved that—

$$\frac{P}{Q} = \frac{1}{8}$$

$$\text{and therefore } P \cdot S_1 = Q \cdot S_2$$

In a similar manner the proposition may be established for any arrangement.

The Inclined Plane. Suppose the power to act along the plane, then if s_2 be the vertical space through which the point of application would be raised if displaced through a distance s_1 in the direction of the plane, we should have by similar triangles

$$\frac{S_2}{S_1} = \frac{\text{height}}{\text{length}}$$

but it has been proved that

$$\frac{P}{Q} = \frac{\text{height}}{\text{length}}$$

$$\text{therefore } P \cdot S_1 = Q \cdot S$$

Suppose in the second place that the power acts horizontally. Here

$$\frac{S_1}{S_2} = \frac{\text{base}}{\text{height}}$$

and it has been proved that

$$\frac{P}{Q} = \frac{\text{height}}{\text{base}}$$

$$\text{therefore } P \cdot S_1 = Q \cdot S_2.$$

From this case of the inclined plane it follows that the principle is also true for the screw.

The cases examined are particular cases of a general principle termed the Principle of Virtual Velocities. The supposed small displacement of a point of applica*tion of a force* in the direction of the force is termed *its virtual velocity*, and the product of the force by

its virtual velocity is termed the virtual moment. The general theorem is stated as follows:—

If any machine is in equilibrium under the action of a system of forces, and we conceive any small displacement of the machine consistent with the connection of its various parts, the algebraical sum of the virtual moments of the forces is zero, and if the sum be zero for all displacements the forces are in equilibrium.

Exercises.

1. When a power of 20 lbs. is applied to lift a weight through 2 in. the power descends through 3 in.; find the weight.—*Ans.* 30 lbs.

2. With a wheel and axle a power of 14 lbs. balances 1 ton; if the power descended through 80 in. through what height would the weight be raised?—*Ans.* ½ in.

3. When there are four movable pulleys in one block, how much string passes through the hands in raising a weight 6 in.? —*Ans.* 4 ft.

4. With a block of three movable pulleys, how much cord would be required for a man to raise himself 30 ft.?—*Ans.* 210 ft.

5. Show that, in order that the pulleys in Fig. 79 may revolve in the same time, the diameters of the lower block must be as the numbers 1, 3, 5, and those of the upper as the numbers 2, 4, 6.

XII.—THE APPLICATION OF TRIGONOMETRY TO THE FOREGOING THEOREMS.

95. If $a\ b\ c$ be sides of a triangle, and A the angle opposite a, then $a^2 = b^2 + c^2 - 2bc \cos A$. Apply this to the triangle A C M (Fig. 20), then

$$R^2 = F^2 + F'^2 - 2FF' \cdot \cos A$$

$$\text{let angle } AMB = \alpha, \text{ then } \cos A = -\cos \alpha$$

$$\text{or } R^2 = F^2 + F'^2 + 2F \cdot F' \cos \alpha.$$

96. The sides of any triangle are proportional to the sines of the opposite angles; hence in the triangle A M C

$$\frac{R}{\sin A} = \frac{F}{\sin ACM} = \frac{F'}{\sin AMC}$$

But since the sine of an angle is equal to the sine of its supplement, if we write (R F) for the angle between R and F we obtain

$$\frac{R}{\sin (FF')} = \frac{F}{\sin (RF')} = \frac{F'}{\sin (RF)}$$

If R′, F, F′ be forces in equilibrium (Fig. 17), then $(R'F') = 180° - (RF)$ and $(R'F) = 180° - (RF)$;

$$\text{therefore } \frac{R'}{\sin (FF')} = \frac{F'}{\sin (R'F')} = \frac{F}{\sin (R'F)}$$

97. The projection of any line of length, l, on a line making with it an angle α, is $l \cdot \cos \alpha$; and the projection on a line at right angles to the former is $l \cdot \sin \alpha$; consequently the result of 32 may be written thus—

$$R^2 = (\Sigma\, P \cdot \cos \alpha)^2 + (\Sigma\, P \sin \alpha)^2$$

Where $\Sigma\,P\cdot\cos a = P_1\cos a + P_2\cos a_2 +$ &c., the sum of the projections of the forces P_1, P_2, P_3, &c. on a line with which they make angles a_1, a_2, a_3, &c.

If a is the angle between the direction of the resultant and the line of reference, since R is the hypothenuse of a triangle, of which X and Y are the sides

$$\tan a = \frac{Y}{X} \text{ or } \frac{\Sigma\,P\cdot\sin a}{\Sigma\,P\cdot\cos a}$$

The conditions that the forces shall be in equilibrium is therefore—

$$\Sigma\,P\cdot\cos a = 0 \qquad \Sigma\,P\cdot\sin a = 0$$

Exercises.

1. Two forces of 25 lbs. and 30 lbs. act on a point at an angle whose cosine is $\frac{1}{3}$; find their resultant.—*Ans.* 45 lbs.

2. The resultant of two forces acting on a point is 536 lbs., and makes angles of 52° 37′ 47″ and 37° 59′ 43″ with the forces; find the forces having given sin 52° 37′ 47″ = ·7947, sin 37° 59′ 43″ = ·6156.—*Ans.* 426 and 330.

3. Two forces of 53 lbs. and 32 lbs. acting on a point have a resultant of 75 lbs.; find the sine of the angle between them. —*Ans.* $\frac{45}{53}$.

4. The resultant of two forces P and Q is equal to P, show that the cosine of the angle between them is $\left(-\frac{Q}{2P}\right)$

5. Three forces, 9, 12, and 15 lbs. respectively, act upon a point and keep it at rest; having given that sin 126° 52′ = $\frac{4}{5}$, find the angles at which the forces are inclined to each other.—*Ans.* 90°, 143° 8′, 126° 52′.

6. One of two components is double the other, and the resultant is equal to half their sum; find the cosine of the angle between them.—*Ans.* $-\frac{11}{16}$.

7. Prove that, if R makes angles of α and β with P and Q,

$$R = P \cos \alpha + Q \cos \beta$$
$$O = P \sin \alpha - Q \sin \beta$$

8. A B C D E is a cord fixed to two points A and E. At B, C, and D, weights P, Q, R, are suspended. Take any vertical line and any point O not in the line. From O draw lines to the vertical line parallel to the parts of the cord, and show that these lines represent the tensions in the parts of the cord, and the parts of the vertical intercepted by them represent the weights.

9. From the preceding construction, show that if the cords at B make angles α, α' with the vertical, at C angles β, β', and at D γ, γ', then

$$\frac{P}{\cot \alpha + \cot \alpha'} = \frac{Q}{\cot \beta + \cot \beta'} = \frac{R}{\cot \gamma + \cot \gamma'}$$

10. When two forces P and Q act at an angle θ, their resultant is $5\sqrt{(P^2+Q^2)}$; but when they act at an angle $90° - \theta$, the resultant equals $3\sqrt{(P^2+Q^2)}$; find θ.—*Ans.* Tan $\theta = \frac{1}{3}$ and $\therefore \theta = 18° 26'$.

11. A line B D bisects an equilateral triangle A B C; forces act along the lines A D, D B, A B as follows: 4 lbs. from A to D, 6 lbs. from D to B, 8 lbs. from A to B: find the resultant.—*Ans.* The resultant passes through the middle point of D B and makes with D B an angle of 31° 45′

(sin 31° 45′ = ·5262139).

12. Three weights, of 4, 5, and 6 lbs. respectively, are suspended over the circumference of a circular hoop, by three strings knotted together at its centre; determine the relative positions of the strings when the hoop supported at its centre remains horizontal. (The moments of forces at two points about the diameter containing the other are equal.)—*Ans.* 97° 11′ (sin = ·99215); 124° 14′ (sin = ·82675); 138° 35′ (sin = ·66153).

13. A and B are two given points in a horizontal line 1 foot apart; to A a string A C is fastened = $\frac{1}{2}$ A B; to B another string is fastened which, passing through a ring at C, supports a weight W at its other extremity; show that B C = ·638; A = 32° 32′ (*cos = ·8429*); *B* = 90 — 2 A.

14. A and B are points 14 inches apart in a vertical wall, to

which are attached the extremities of a cord A C D B. At C a weight of 11 lbs. is attached, and at D another weight W. A C = 6 in., C D = 7 in., and D B = 8 in. What must be the weight of W in order that C D may be horizontal?—*Ans.* 3 lbs.

Parallel Forces—Couples.

98. A couple consists of two equal parallel forces acting in opposite directions.

The perpendicular distance between the directions of the forces is the arm of the couple. The tendency of the couple is to turn the arm about an axis perpendicular to the plane of the couple at the middle point of the arm. This line is usually termed the axis of the couple. The effect of the couple will not be altered by any change which leaves the moment about the axis the same.

Take any point in the plane of the couple P P, and let x_1 be the distance of the point from the direction of one force, x_2 the distance from the other. Also let a be the arm of the couple.

If the point be without the couple, the moment of the couple about the point

$$= \text{P}\, x_1 - \text{P}\, x_2 = \text{P}\,(x_1 - x_2) = \text{P}\cdot a.$$

If the point be between the forces, the moment about the point

$$= \text{P}\, x_1 + \text{P}\, x_2 = \text{P}\,(x_1 + x_2) = \text{P}\cdot a.$$

The moment of a couple about all points in its plane is therefore constant, and two couples are equivalent when their moments are equal; hence the following proportions will be easily established:—

1. A couple may be turned in its own plane through

any angle about any point in its own arm, without altering its statical effect.

2. A couple may be moved parallel to itself without altering its statical effect.

3. Two couples are equivalent if their moments are equal and in the same direction.

4. The resultant of any number of couples acting in the same plane or in parallel planes is a couple the moment of which is the algebraical sum of the moments of the couples.

99. *To find the resultant of two couples not in the same plane.*

For the given couples substitute equivalent couples, P P, Q Q, having for common arm a portion M N of the line of intersection of the two planes. Find the resultant R of the forces P and Q at M and at N. Since the forces at M are equal and parallel to those at N, the resultant at M is equal and parallel to that at N, and the resultant of the two couples is a couple whose moment is R $\times$ M N.

Now, since all the couples have a common arm, their moments are proportional to the forces.

At any point whatever O, draw a line Op perpendicular to the plane of P P, and representing, according to a certain scale, the moment of P $\times$ M N. From the same point draw O q perpendicular to the plane Q Q, and representing, on the same scale, the moment Q $\times$ M N. Complete the parallelogram O p q r, it will be similar to the parallelogram M P Q, and therefore O r will be the perpendicular to the plane R R, and will represent on *the chosen* scale the moment of the resultant couple.

Hence, if two straight lines, drawn from the same

point, have the directions of the axes of two couples, and are proportional to the moments of the couples, the diagonal of the parallelogram on these lines drawn through the point has the direction of the axis and the magnitude of the moment of the resultant couple.

Hence the laws of the composition and resolution of couples are similar to the corresponding laws of forces, the axis of the couple corresponding to the direction of the force, and the moment of the couple to the magnitude of the force. For example, if L and M be the moments of the component couples, G the moment of the resultant, and θ the angle between their planes, then $G^2 = L^2 + M^2 + 2\,LM\cos\theta$.

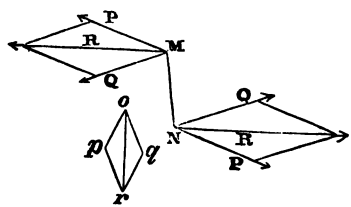


Fig. 103.

EXERCISES.

1. Three equal forces act along the sides of an equilateral triangle taken in order; show that they are equivalent to a couple whose moment is $\frac{\sqrt{3}}{2} \cdot P \cdot a$.

2. Three parallel forces, P, 2 P, and 3 P, act at the angular points of an equilateral triangle; determine the lines along which they act when there is equilibrium.

3. Three *forces* act along the sides of a triangle and are pro-

portional to the sides; show that they are equivalent to a couple whose moment is equal to twice the area of the triangle.

The Reactions of Surfaces.

100. When the forces act at different points in a body, the following summary of facts should be borne in mind:—

1. When two forces are in equilibrium they must act in the same straight line.

2. When three forces not parallel are in equilibrium their lines of direction must meet in a point.

3. Reactions of surfaces are perpendicular to the surfaces.

4. When there are two unknown forces an equation may be found containing only one, either by taking moments about some point in the other, or by resolving the forces in a direction at right angles to this other.

5. When there are three unknown forces two equations may be found, from which a force not required is excluded by resolving in a direction perpendicular to that of the force, and by taking moments about a point through which this unknown force passes.

Example. A uniform beam A B is placed with one end A inside a smooth hemispherical bowl and a point D in it resting on the edge of the bowl; find the inclination of the beam.

Draw the diameter A N and join N D; then A D N being the angle of a semicircle is a right angle. The reaction R of the edge of the bowl at D will therefore be in the direction D N, and the reaction R′ of the bowl at A in the direction A N. A vertical line therefore drawn from N will pass through G the C. G. of the beam.

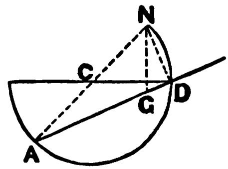


Fig. 104.

Let $CAD = \theta$, then $CDA = \theta$, and $CDN = 90^\circ - \theta$

$NCD = 2\,\theta$, and $CNG = 90^\circ - 2\,\theta$.

Let $CA = r \quad AG = a$.

Resolve the forces horizontally.

Take a line on CA to represent R' and its projection on $CD = R' \cos C = R' \cos 2\,\theta$.

Take a line on DN to represent R and its projection on $CD = R \cdot \cos CDN = R \cdot \sin \theta$.

Hence $R \cdot \sin \theta = R' \cdot \cos 2\,\theta$.

Now take moments about G.

The perpendicular from

$$G \text{ on } AN = AG \sin \theta;$$

hence the moment of

$$R' = R'a \sin \theta.$$

The moment of

$$R \text{ about } G = R \cdot GD;$$

but $GD = AD - AG = 2r \cos \theta - a$.

Hence $R'a \cdot \sin \theta = R\,(2r \cos \theta - a)$.

Multiply this and the previous equation together, and RR' go out; therefore

$$a \sin^2 \theta = \cos 2\,\theta\,(2r \cos \theta - a)$$

hence reducing

$$4r \cos^2 \theta - a \cos \theta - 2r = 0.$$

$$\text{Cos}\,\theta = \frac{a}{8r} \pm \sqrt{\left(\frac{a^2}{64r^2} + \frac{1}{2}\right)}$$

EXERCISES.

1. A sphere of weight W rests on two planes inclined at angles α and β to the horizon; find the normal pressures on the planes.—*Ans.* On the first $\dfrac{W \sin \beta}{\sin (\alpha + \beta)}$; on the second $\dfrac{W \sin \alpha}{\sin (\alpha + \beta)}$

2. A wire is bent into the form of a quadrant, placed with a radius O A horizontal and a radius O B vertical; over a small pulley at A a cord is passed supporting two weights, one P hanging freely, and the other 2P in the shape of a ring capable of sliding along the wire; find the arc from A to 2P.—*Ans.* $65° 4'$. (Cos. $32° 32' = \cdot 84307$.)

3. A uniform beam rests on two planes inclined at angles α and β to the horizon; find the inclination of the beam to the horizon, and the pressures on the planes.—*Ans.* $\tan \theta = \dfrac{\sin (\beta - \alpha)}{2 \sin \alpha \sin \beta}$; $R = \dfrac{W \sin \beta}{\sin (\alpha + \beta)}$; $R' = \dfrac{W \sin \alpha}{\sin (\alpha + \beta)}$

4. If in Figure 104 $2a = 3r$, find A D.—*Ans.* $1 \cdot 838 \times r$.

5. Two spheres rest upon two smooth inclined planes and press against each other; determine the inclination to the horizon of the line joining their centres.—*Ans.* If α β be the angles of the planes, W W′ the weights of the spheres, and θ the inclination of the line joining the centres $\tan \theta = \dfrac{W' \cot \alpha - W \cot \alpha'}{W + W'}$

6. A uniform beam A B whose weight is 50 lbs. rests with one end B on the top of a vertical wall the height of which is half the length of the beam, the other end A is on the horizontal plane A D, and is prevented from sliding by a string D A equal to $\frac{3}{4}$ of A B; find the tension of the string.—*Ans.* 14·4 lbs. (Resolve horizontally and take moments about A.)

7. A beam A B 10 ft. long, weighing 10 lbs. a foot, has a string A C 10 ft. long attached to the lower end A, and tied to a ring C, and at the other a string which passes through the

ring and bears a weight of 50 lbs.; find the inclination of the beam.—*Ans.* $\cos. \theta = \sqrt{\frac{7}{8}}$. (Take moments about A and resolve perpendicularly to A C.)

8. A rod A B weighing 1 cwt., whose C. G. is 5 ft. from the end A, rests against a smooth vertical wall, and on a prop which is 3 ft. from the wall; find the inclination of the rod and the pressures on the wall and prop.—*Ans.* $\cos \theta = \sqrt[3]{\cdot 6}$, the pressure on the prop $= \cdot \sqrt[3]{1 \cdot \dot{6}}$, and on the wall $\sqrt{(\sqrt[3]{2 \cdot \dot{7}} - 1)}$.

9. At what point of a tree must a rope of given length a be fixed so that a force at the other end may have the greatest tendency to overturn the tree?—*Ans.* At a height $\frac{a}{\sqrt{2}}$.

Mechanical Powers.

101. All kinds of levers, bent and straight, may be regarded as bodies capable of moving about a fixed axis; and the condition of equilibrium is, therefore, that the sum of the moments of the forces is zero.

In the general case of the inclined plane (Fig. 82), the triangle O H E, which has its sides proportional to the forces W, F and R, has one angle D O E, equal to α, for it is contained by sides perpendicular to the sides of the angle A. If the force F make an angle ϕ with the perpendicular to the plane, the angle H O E $= 180° - \phi$.

$$\text{Now } \frac{F}{W} = \frac{\sin \text{ OEH}}{\sin \text{ HOE}} = \frac{\sin \alpha}{\sin \phi}$$

$$\frac{R}{W} = \frac{\sin \text{ OHE}}{\sin \text{ HOE}} = \frac{\sin (\phi - \alpha)}{\sin \phi}$$

When friction is taken into account we have an additional force equal to μ R, acting up or down the plane

according as the body is on the point of moving down or up. By resolving the forces along the plane, and perpendicular to it, we obtain two equations to determine F and R.

Thus, along the plane

W gives W $\sin \alpha$ down the plane,

Friction gives $\pm \mu$ R up or down

and F gives $-$ F $\sin \phi$ up the plane.

And perpendicular to the plane

W gives W $\cos \alpha$ downwards,

R gives $-$ R upwards,

F gives $-$ F $\cos \phi$ upwards.

$$\text{Hence } W \sin \alpha \pm \mu R - F \sin \phi = 0,$$

$$\text{and } W \cos \alpha - R - F \cos \phi = 0.$$

Example. A mass of wrought iron weighing 1C0 lbs. would just rest on a plane of oak inclined at an angle of 32°; if it rest on a plane inclined 20°, find the greatest pressure P which may act on it at an angle of 10° to the plane, without moving it up the plane.

The tables give the coefficient of friction $= \tan 32° = \cdot 625$

$$\sin \alpha = \sin 20° = \cdot 342\,;\ \cos \alpha = \cdot 94,$$

$$\sin \phi = \sin 80° = \cdot 985\,;\ \cos \phi = \cdot 174.$$

Hence the above equations become

$$100 \times \cdot 342 - \cdot 625\,R - F \times \cdot 295 = 0,$$

$$100 \times \cdot 94 - R - F \times \cdot 174 = 0.$$

Eliminating R we have F $= 28$ lbs.

EXERCISES.

1. Two uniform rods forming a right angle are suspended from the right angle. If one be 18 in. and the other 24 in. long, find the inclination α of the shorter to the horizon. — *Ans.* $\tan \alpha = 2\frac{1}{18}$.

2. In the above, if the shorter rod $= 2m$, and the longer $2n$, show that

$$\tan \alpha = \frac{m^2 + 2mn}{n^2}$$

3. A lever A B, of weight W, moving about a hinge at A, is supported by a cord tied at B, and passing over a pulley C. To the other end of the cord is attached a weight = 3 W. The length of the beam, the distance of the hinge from the vertical plane containing the pulley, and the height of C above the plane containing A, are equal to one another, show that if the beam makes with the horizon an angle α, and the cord makes with the vertical an angle β, then

$$\cos(\beta + \alpha) = \tfrac{1}{6} \cos \alpha = \cos \beta - \sin \beta.$$

4. A B C is a bent lever, forming an angle of 150°; A B is 21 in., B C 35 in. long; weights of 28 and 24 lbs. are suspended from A and C; find the inclination of the arms to the horizon. —*Ans.* The shorter arm 18° 22′ above (sin = ·315), and longer arm 48° 22′ below (sin = ·7474).

5. A weight W is supported on a smooth inclined plane by three forces, each equal to $\frac{1}{3}$ W, which act one vertically upwards, another horizontally, and the third parallel to the plane; find the inclination of the plane.—*Ans.* $\tan \frac{\theta}{2} = \cdot 5$ and therefore $\theta = 53° \; 7' \; 48''$.

6. A uniform beam A B, of weight W, rests with one end A on a horizontal plane A C, and the other end on a plane C B, whose inclination to the horizon is 60°. If a string C A equal to C B prevent the beam from sliding, what is the tension?— *Ans.* $\frac{\sqrt{3}}{4}$ W.

(*Take moments* about A and resolve horizontally.)

7. A uniform beam A B rests with the upper end B on a prop, and the lower end A attached to a string, which after passing over a pulley C bears a weight equal to one-third that of the beam; find the position of equilibrium.—*Ans.* If θ be the inclination of the beam, and $(\theta + \beta)$ that of the string A C, then $\cos\theta = \frac{2}{3}\sin\beta$.

8. Weights of 40 lbs. and 50 lbs. are attached at the extremities A B of a light rigid rod A B, 17 ft. long, which is supported by a cord 19 ft., tied at both ends of the beam, and passing over a small pulley O; find the position of equilibrium.—*Ans.* $A\,O = 10 \cdot \dot{5}$ ft.; $B\,O = 8 \cdot \dot{4}$ ft.; $\sin \angle O = \frac{720}{1805}$, and the vertical through O bisects the angle O.

9. If on an inclined plane the pressure, force and weight be as the numbers 28, 17, and 25, find the inclination of the plane to the horizon, and of the force to the plane.—*Ans.* 36° 52′ 12″.

$$(\sin = 6) \text{ and } 28° \, 4' \, 15'' \; (\cos = \tfrac{15}{17})$$

10. A weight of 100 lbs. is sustained on a rough plane, inclined at an angle, and by a force F inclined at an angle β to the plane; if the greatest angle at which the body would rest is 45°, find the limits between which F must lie.—*Ans.* between

$$50\left(\frac{\sin\alpha + \cos\alpha}{\cos\beta}\right) \text{ and } 100\left(\frac{\sin\alpha - \cos\alpha}{\cos\beta - \sin\beta}\right)$$

11. A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the centre of gravity of the ladder is at $\frac{1}{3}$ the length from the foot. The coefficient of friction for the ladder and plane is $\frac{1}{3}$, and for ladder and wall $\frac{1}{4}$. If a man whose weight is half the weight of the ladder ascend it, find to what height he will go before the ladder begins to slide.—*Ans.* $\frac{19}{39}$ of the length.

(Resolve horizontally and vertically, and take moments about the foot.)

DYNAMICS.

I.—MOTION.

1. The examples of motion which constantly recur to our observation are very varied. Sometimes a body moves in a straight line, as when it falls freely to the earth. Sometimes it describes a curve, as when it is projected in a direction not vertical. Sometimes the body turns and retraces its path, as is the case with the ball of a pendulum. The velocity with which the body moves, and the nature of the forces which produce motion, are other elements admitting of endless variation. It will evidently be necessary, therefore, to consider some of these circumstances of motion apart from the others. For example, it will be convenient to consider, first, the conditions of motion of a body independently of its size and shape, and, making an abstraction of these properties, to consider the motion of a very small particle of matter or a material point.

2. Before investigating the effect of forces in producing or changing the motion of bodies, it will be convenient to consider the motion of a point independently of the forces which produce it. This branch of mechanics has been termed *kinematics*; and then *the term dynamics* is restricted to the science which

investigates the relations which exist between the forces and the motion they produce.

3. When a moving point or particle is in motion, the line containing all its successive positions is termed the *path* of the point. The path may be a straight line or a curve.

4. The motion of a particle is not sufficiently defined by the path it describes. The particular points in the path occupied by the moving particle at different instants must also be known. For example, railway time-tables determine the motion of a train when they give the time at which the train arrives at the several stations.

Suppose a particle to move on a given path M N (Fig. 105), its motion is defined in the following

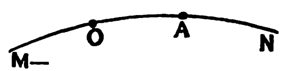


Fig. 105.

manner. A fixed point O is chosen arbitrarily to serve as the origin from which the lengths of the arcs of the curve are measured. The position of a point A is indicated by the length of the arc O A. Arcs measured in one direction O N are considered positive, and arcs measured in the opposite direction O M are negative.

5. The motion of a point is said to be *uniform* when the point passes over equal spaces in equal times.

In the case of uniform motion the velocity of the moving point is constant, and is measured by the length of path passed over in a unit of time. This length is usually expressed in feet, and the time in seconds. Frequently, however, other units are chosen; thus, a *train may proceed* with a speed of 40 miles an hour,

a ship may sail with a speed of 10 knots an hour. Velocity expressed in other units may, however, be readily reduced to feet per second. For example,

$$1 \text{ mile an hour} = 1\tfrac{7}{15} \text{ ft. per sec.}$$

The velocity is *variable* when the lengths of the path described in equal times are not equal.

6. The length of path described in a certain time divided by the time is termed the *mean* velocity for that portion of the path.

7. Variable velocity at any instant is measured by the mean velocity for an infinitely small space commenced at that instant.

It is the space the body would describe in a unit of time if from that particular instant the velocity remained constant. For example, a horse may travel from one place to another with a variable velocity; but we may say that at a particular instant he is running at a speed of 20 miles an hour. We mean that for a small distance he moves with a speed which, if maintained for an hour, would carry him over 20 miles.

Direction of Velocity.

8. Let A A′ be points near together on the path. The arc A A′ differs but little from a straight line. Let the straight line A A′ be produced to B. The direction A B has two points in common with the curve. When the points are indefinitely near, the line is called in geometry a tangent to the curve. This line is the direction of the motion of the particle at A. Hence the *direction* of the velocity of a moving particle at any *point is the tangent* to its path at this point.

The magnitude of the velocity will be represented by taking on A B a length A C, equal to the space which the particle would describe in a unit of time if it traversed the line A B with a uniform velocity equal to that which it had at the point A.

Thus, if θ be the time the particle takes to pass over the small arc $a\,a'$, A C =

$$\frac{\mathrm{A\,A'}}{\theta}$$

Hence the tangent A C represents the velocity in direction and magnitude.

If the path be a straight line the direction of the velocity is that of the path.

Graphic representation of Motion.

9. It is frequently very useful to represent the properties of the motion of a point by means of lines.

The motion of a particle along its path is determined when its distance from a fixed point on its path is known for every instant of time, and this is always the case when we can express the relation between the space S and time t. This relation may be represented by a line.

Let M N be the path (Fig. 105), O the fixed point from which arcs are measured, A the position of the particle at the end of time t; O A will be the corresponding value of S. Draw two straight lines at right angles A X, A Y (Fig. 106); choose a unit of length to represent a unit of time; set off along A X a line A E to represent, according to the scale chosen, the time t.

At the point E draw a straight line E F parallel to

A Y; take a certain scale of lengths to represent units of space, and set off along E F a distance E F = s.

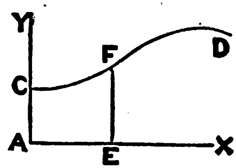


Fig. 106.

If the construction be repeated at different instants, for every value of t set off along A X there will be a corresponding value of s set off on a line parallel to A Y; and to every point A in the path of the moving particle will correspond a point F in the plan. The successive points will furnish a continuous line, which will represent the law of movement. We will term this line the *curve of spaces.* Lines measured on A X are termed abscissæ, and lines parallel to A Y are called ordinates.

The arc s may be positive or negative according as A is to the right or left of o. The positive arcs are represented in the plan by ordinates above C X, and the negative arcs by ordinates below. The point C corresponds to $t = o$, that is, to the instant from which the time is reckoned.

Time anterior to this instant will be considered negative, and will be represented by lines from A to the left.

With these conventions, all the circumstances of the motion of the particle may be determined by an inspection of the curve of spaces. Suppose, for example, *that the figure (107)* represents the curve of spaces for *a particle moving* in a straight line.

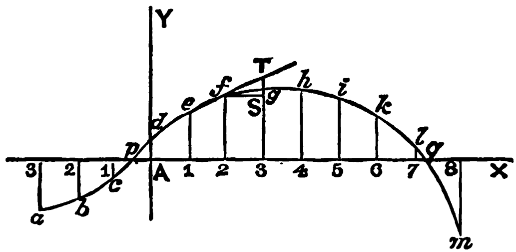


Fig. 107.

p represents the zero point o, from which the spaces are measured. The particle returns to o in the time represented by $p\,q$, that is, in 8 seconds. The instant from which the time is taken is not that at which the particle is at o, but at an interval represented by A p after this time.

Let M N be the straight line along which the particle moves (Fig. 108), and let the point o be that from which

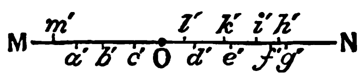


Fig. 108.

its distance s is measured. Then o corresponds to the points p and q on the curve of spaces.

It is easy to find on the path M N the position of the particle for every other value of the time t comprised between the limits of the curve, that is, from — 3 to + 8 seconds, or, in all, a period of 11 seconds.

Take on the left of O,

$O\,a' =$ the ord. at a

$O\,b' =$,, b

$O\,c' =$,, c

Take on the right of O,

$O\,d'$	= the ord. at	d		$O\,h'$	= the ord. at	h
$O\,e'$	= ,,	e		$O\,i'$	= ,,	i
$O\,f'$	= ,,	f		$O\,k'$	= ,,	k
$O\,g'$	= ,,	g		$O\,l'$	= ,,	l

The points a' b' to m' are the successive positions of the particle at the end of each second. The points p and q at which the curve intersects the axis A X give the instants at which the particle passes the point O. The greatest ordinate is that at g corresponding to the time 3 seconds. This indicates that the particle changes its direction at g' and returns upon its path.

The velocity of the particle at any point may also be found from the curve of spaces. Let it be required, for example, to find the velocity of the particle at the point f' of its path. At the point f draw a tangent f T to the curve, and a line f S parallel to A X cutting the ordinate at g in the points T and S; then S T represents the velocity at the point f'.

Now if from the point f the velocity were uniform, the space passed over divided by the time would be constant. Let T' be a point on the curve infinitely near to f, T is also a point on the tangent. Let f S' represent the infinitely small interval of time in which the particle passes over the infinitely small space S' T' (Fig. 109); then

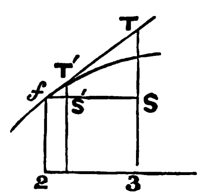


Fig. 109.

$\frac{S'T'}{fS'}$ is the velocity, but $\frac{S'T'}{fS'} = \frac{ST}{fS}$; hence

S T is the space passed through in the time f S. But f S represents the unit of time, consequently S T is the space through which the body would pass in the unit of time, provided it kept throughout this time the velocity it had at the commencement.

If this construction be made at the point g where the tangent to the curve is parallel to A X, it gives the velocity zero, indicating that the particle at g' in its path stops for an instant, and the motion then becomes retrograde. At a point k beyond the point g the tangent would cut the ordinate below the parallel to A X, showing that the velocity is then negative. This is the case whenever the motion is retrograde.

Curve of Velocities.

10. Suppose that the curve of spaces B C D E F has been traced.

The velocity of the moving point on its path at a particular instant is given by a certain length parallel to the axis A Y. This line represents on the scale of lengths a space described in a unit of time with the same velocity.

We may take these lengths representing velocity to construct a second curve such that the ordinates indicate the velocity of the moving point just as the ordinate of the first indicated the arcs described.

Let us take any ordinate M P (Fig. 110) and construct the velocity Q R corresponding to the instant $t =$ C M. *Measure on* M P a length M $p =$ Q R. Repeat this con-

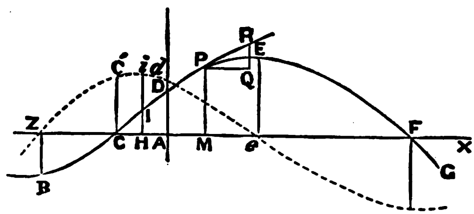


Fig. 110.

struction for a number of points, and a line is obtained $b\,c\,d\,e\,f\,g$, which will be the curve of velocities.

The inspection of this curve shows that the velocity is positive from the point b to the point e, that is to say, from the instant $t = -\text{A}\,b$ to the instant $t = +\text{A}\,e$.

During this period the movable point displaces itself in the positive direction on its path. The points b and e are the projections on the axis of the points B and E on the curve of spaces, corresponding to a maximum or a minimum value of the ordinate. At the instant indicated by $t = \text{A}\,e$ the direction of motion changes. The velocity is a maximum at the point i, that is to say, at the instant $t = -\text{A}\,\text{H}$. At that instant the curve of spaces gives $\text{S} = +\text{H}\,\text{I}$, and at the point I is a change of inflexion. When t is greater than $\text{A}\,e$ the velocity is negative and the motion retrograde.

We have seen how the curve of velocities can be deduced from the curve of spaces; we will now show how the inverse problem may be solved.

To find the space passed over in a given time when the law of velocity is known.

11. Construct the curve of velocities thus—Upon a straight line commencing at a fixed point, measure off distances to represent units of time, at each interval raise perpendiculars proportional to the velocities at the corresponding instant, and lastly draw a line through the extremities of these perpendiculars. We may consider the movement of the point to commence at any instant whatever. For example, let the point be supposed to start at the instant defined by the abscissa $t = \text{A}\,a$. Let us divide the axis of the time, com-

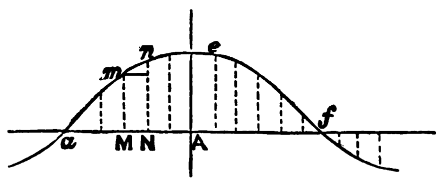


Fig. 111.

mencing at a, into equal parts which are infinitely small. Each part will represent a time θ indefinitely short. At the points of division draw the ordinates of the curve which will represent the successive values of the velocity at the instants defined by the following values of the time—

$$(-\text{A}\,a + \theta), (-\text{A}\,a + 2\,\theta), (-\text{A} + 3\,\theta), \&c.$$

Let v be the ordinate for a certain value t of the time. This ordinate v represents the velocity at that instant, that is, — the space that would be described if the velocity continued uniform during a unit of time.

The space described by the moving point on its path *during the time* θ, will be the product $v\,\theta$. . Now this

product is represented on the figure, for if M $m = v$, and M N $= \theta$, it is the area of the rectangle, m N. The smaller we make θ the more nearly this rectangle approaches the area comprised between the arc of the curve, the axis of the time and the two ordinates M m, n N. Each one of these infinitely small areas represents therefore the space described by the moving point on its path during the same time θ, commencing at the instant defined by the corresponding abscissa.

The negative values of the velocity indicate retrograde motion; we ought, therefore, to consider as negative the areas corresponding to negative ordinates. Hence, taking the signs as thus defined, the algebraical sum of all these elements of surface from any point a to any point m, that is to say, the area of the curve of velocities will represent the distance of the moving point from the starting point $t =$ —A a to the instant $t =$ + A M, or in other terms—

The successive areas of the curve of velocities starting from any point whatever, a on the axis A X, will represent in magnitude and in sign the total and the successive displacements of the moving point on its path, commencing from the point which it occupies at the instant defined by the abscissa at the point a.

The problem, therefore, resolves itself into finding the areas of curves. Whenever the line is one of those for which geometry furnishes a method of finding the area, as, for example, a straight line, circle, parabola, &c., we can readily construct the curve of velocities from the line of spaces. When the curve does not belong to a known type we must resort to approximations.

Curve of Accelerations.

12. The velocity is a magnitude which generally varies with the time, and the rate of variation of the velocity is a new property of the motion termed the acceleration. It may be uniform or variable. *When the velocity increases uniformly the acceleration is the velocity added in a unit of time.*

When the increase of velocity is not uniform, the acceleration at every point is the velocity which would be added in a unit of time, if the velocity continued to increase at the same rate throughout this time.

We might represent the law of acceleration, as we have indicated the law of velocity, by a curve of acceleration in which the ordinates are proportional to the accelerations.

Such a curve may evidently be deduced from the curve of velocity in precisely the same way as this curve was deduced from the curve of spaces, namely, by drawing tangents at the different points in the curve. Similarly, as the curve of spaces can be obtained from the areas of the curve of velocities, so the curve of velocities can be found by taking ordinates proportional to the areas of the curve of acceleration. The investigation of this curve may be left with the student.

Uniform Motion.

13. When the motion is uniform the velocity v is the space passed over in every second of time; hence in 2 seconds the particle moves over $2\ v$ ft., in 3 seconds over $3\ v$ ft., and so on. Hence the motion is defined by the equation

$$S = vt.$$

The line of spaces is a straight line, for the ordinates increase in a constant ratio. The line of velocities has its ordinates constant and is a straight line parallel to the axis A x at a distance $= v$.

Composition of Velocities.

14. Suppose a particle at O to have imparted to it simultaneously a velocity v along OX and a velocity v'

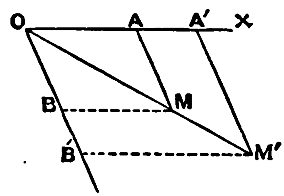


Fig. 112.

along OY (Fig. 112). Let M be a position of the particle at time t, and suppose MA drawn parallel to OY meeting OX in A, then the motion may be thus described. The point M moves along the line AM parallel to OY with velocity v' while this line is displaced parallel to itself, so that its extremity A moves along OX with velocity v.

L

Therefore $OA = vt$ and $AM = v't$.

$$\text{Hence } \frac{OA}{AM} = \frac{v}{v'}$$

Let M' be the position of the particle at time t'.

Then also $OA' = vt'$ and $A'M' = v't'$.

$$\text{Hence } \frac{OA}{AM} = \frac{OA'}{A'M'}$$

Suppose now the points M and M' to be each joined with O, then the above proportion shows that the triangles OAM, $OA'M'$, which have angles A and A', equal are similar, and therefore that the angles AOM, $A'OM'$ are equal, and OMM' is a straight line.

The same triangle gives the proportion

$$\frac{OM}{OM'} = \frac{OA}{OA'} = \frac{vt}{vt'} = \frac{t}{t'}$$

showing that the space described by the point M is proportional to the time; hence M moves in a straight line with uniform velocity.

To determine this velocity let t in the above equation be the unit of time, then OA represents v, AM represents v', and OM the resultant velocity of M. Through M draw MB parallel to OX, then $OB = AM \quad v'$ and it is evident that the resultant velocity is the diagonal of the parallelogram constructed on OA, OB, which represent the component velocities.

If two velocities imparted simultaneously to a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant velocity will be represented in magnitude and direction by the diagonal of this parallelogram drawn through the particle.

From this proposition it follows that when a point *moves in a straight* line with uniform velocity, the pro-

jection of the point on any straight line whatever moves with uniform velocity.

It follows also that any uniform velocity may be considered as resulting from two simultaneous velocities, the components being represented by the sides of a parallelogram having the line representing the resultant velocity for diagonal.

Exercises.

1. A body moves uniformly over 5 miles in half an hour; determine its velocity.—*Ans.* $14 \cdot \dot{6}$.

2. A body moves at the rate of 3 miles a quarter of an hour; determine its velocity.—*Ans.* 17·6.

3. The equatorial diameter of the earth is 41,847,000 ft., and the earth makes one revolution in 24 hours; determine the velocity of a point on the earth's equator.—*Ans.* 1526.

4. Find the velocity of a point a, latitude at 60°.—*Ans.* 763.

5. A body moves with a velocity 15; how many miles will it pass over in one hour?—*Ans.* $10\frac{5}{22}$.

6. Which velocity is the greater, 3 ft. in half a second, or 70 yards a minute?

7. Compare the velocities of two points which move uniformly, one through 420 ft. in 1 minute, and the other through $2\frac{1}{2}$ yards in $\frac{5}{8}$ of a second?—*Ans.* The second is $1\frac{5}{7}$ of the first.

8. A railway train travels over 100 miles in 2 hours; find the average velocity referred to ft. and seconds.

9. A body starts from a point and moves uniformly along a straight line at the rate of 25 ft. per second. At the end of 20 seconds another body starts from the same point after the former body, and moves uniformly at the rate of 20 miles an hour. Find when and where the second body overtakes the first.—*Ans.* The second moves for $115\frac{5}{13}$ secs. and $\frac{25}{39}$ of a mile.

10. Two bodies start together from the same point, and move uniformly in directions at right angles to each other; one body

moves at the rate of 4 ft. per second, and the other at the rate of 3 ft. per second; find the distance between them at the end of 7 seconds.—*Ans.* 35 ft.

11. A mill sail $8\frac{3}{4}$ yards long goes round uniformly 12 times in a minute; find the velocity of the extremity of the sail .—*ns.* 33 ft. per sec.

12. Two messengers start at the same time, and in the same direction, from places 3 miles apart; one travels at the rate of $4\frac{1}{2}$ and the other at the rate of 6 ft. per second; when will the second overtake the first?—*Ans.* In 2 hrs. 56 mins.

13. M, N, and P are places on the same road. A traveller A starts from M at the same time as another traveller B leaves N. They reach P together, and it is found that they have walked together 30 miles; that B would walk at his rate from P to M in 9 hours, and A from P to N in 4 hours; find the distance M N.—*Ans.* 6 miles.

Motion in a straight line with Uniform Acceleration.

15. The velocity is said to be uniformly accelerated when it is increased in equal intervals of time by a constant quantity. Let f be the increase of velocity for each second, then if v be the velocity at the commencement of the time, the velocity at the end of

$$\begin{aligned} \text{the 1st sec.} &= v + f \\ \text{2nd sec.} &= v + 2f \\ \text{3rd sec.} &= v + 3f \\ &\cdots\cdots\cdots \\ \text{at time } t &= v + ft \end{aligned}$$

Construct the line of velocities (Fig. 113). Let A C = v, and let A a, $a\,b$, $b\,c$, &c. represent equal intervals of time T. The ordinates at the successive intervals will be obtained by adding to the preceding ordinate

the constant quantity f T, represented by $a_1 a'$, $b_1 b'$, &c. If A B represent a time t, the ordinate B D $=$ V $+ f t$. C D is therefore a straight line.

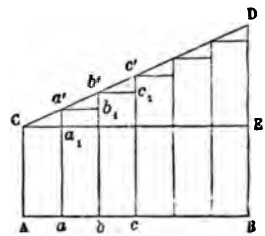


Fig 113.

The space passed over in the time t will be represented by the area of the figure A B D C, that is, by the rectangle A B E C + the triangle C E D, or A C $\times$ A B $+ \frac{1}{2}$ E D $\times$ C E.

$$\text{Hence } s = v\, t + \tfrac{1}{2} f t^2.$$

If the point start from rest V $=$ o, and the space is represented by the triangle C E D, hence $s = \frac{1}{2} f t^2$.

The area A B D C may also be written thus—

$$AB \times \tfrac{1}{2}(AC + BD).$$

Hence the space passed over by a particle moving with uniform acceleration is found by multiplying half the sum of the initial and final velocities by the time.

When the velocity is *diminished* by a quantity which is constant for equal intervals of time, the motion is said to be *uniformly retarded*, and the acceleration is considered negative. When this is the case D will fall below E, and the equations of motion become

$$v = V - f t, \quad s = V t - \tfrac{1}{2} f t^2.$$

The student should consider well the representation of the motion in Fig. 113. The area A B E C represents the space which would be passed over in the time t, with initial velocity only. The little triangles C $a_1 a'$, $a' b_1 b'$, *&c.*, represent the spaces passed over in each

second, and that of the triangle C E D, the space in time t, in consequence of the acceleration.

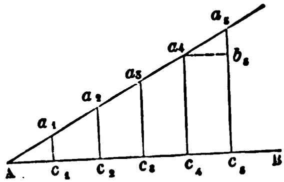


Fig. 114.

If the initial velocity be zero, then the figure becomes a triangle (Fig. 114). If Ac represent the time t, ca the velocity ft, then the space $= \frac{1}{2} Ac \times ca = \frac{1}{2} ft^2$.

Uniform Motion in a Circle.

16. Let a point B move in a circumference, and let B′ be the position of the point B at the commencement of the time, while the point B describes the arc B B′ the radius O B′ describes the angle B′ O B. The revolution of the radius marks what is called the angular velocity of the point B. When the radius describes equal angles in equal times *the angular velocity* is uniform and is equal to the angle described by the radius in a unit of time.

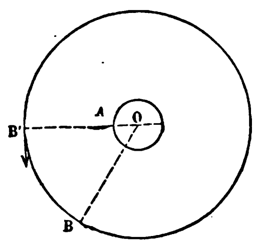


Fig. 115.

When the radius does not describe equal angles in equal times the angular velocity at any point is the angle which would be described by the radius if it continued to revolve for a unit of time with a uniform angular velocity. If the point B moves with a uniform *linear velocity,* equal arcs will be described in equal

times, and since equal arcs correspond to equal angles at the centre, the angular velocity will also be uniform. The angular velocity is not sufficient to define the motion of a point, for if a point A describe the smaller circumference in the same time as the point B describes the larger, the angular velocity will be the same, but the linear velocity will be proportional to the circumference or the radius. Hence, if the radius and angular velocity be given, the linear velocity may be found, and *vice versa.*

Let r be the radius, v the linear velocity, a the angular velocity.

Let the point describe the circumference c in a time t, then

$$t\,v = c$$

and $t\,a = 360°$. Hence $\dfrac{v}{a} = \dfrac{c}{360°} = \dfrac{\pi r}{180°}$

For example, if the angular velocity be 60° per second and radius 3 ft., we have

$$\frac{v}{3} = \frac{\pi 60}{180}$$

$$\text{or } v = \frac{22\,ft}{7} = 3\tfrac{1}{7}\text{ ft.}$$

The tangential acceleration in this case is nothing, for the velocity is uniform.

17. Let the point B (Fig. 116) start from B′, and

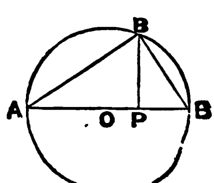


Fig. 116.

let the line B P be drawn perpendicular to the diameter B′ A, consider the velocity of the point P. At the commencement of the motion B moves in the direction of the tangent at B′, and the velocity of P is zero. This velocity increases until P reaches O, when B and P move in parallel directions with the same velocity. Hence the acceleration of P is not zero. The acceleration of P, corresponding to an infinitely small arc B B′, is termed the *normal acceleration.*

Let f be this normal acceleration, and let the point B describe the arc B B′ in the time t;

$$\text{then} \quad PB' = \tfrac{1}{2} f t^2$$

$$\text{and the arc} \quad BB' = v\,t$$

but in the right-angled triangles A B B′ and B P B′

$$\frac{AB'}{BB'} = \frac{BB'}{PB'}$$

$$\text{therefore} \quad PB' = \frac{BB'^2}{AB'}$$

Now, since B B′ is a very small arc, the chord B B′ may be considered equal to the arc B B′

$$\text{hence} \quad \tfrac{1}{2} f t^2 = \frac{v^2 t^2}{2r}$$

$$\text{and} \quad f = \frac{v^2}{r}.$$

Exercises.

1. A body had at one instant a velocity of 15 ft. per second, and 3 seconds later a velocity of 30 ft. per second; find the acceleration.—*Ans.* 5 ft.

2. A point has a velocity of 60 ft. per second after it has been moving 5 seconds from rest; find the acceleration.—Ans. 12 ft.

3. The acceleration is 8 ft. per second, and the initial velocity 11 ft. per second; find the space passed over in 8 seconds, and the velocity at the end of the time.—*Ans.* 344 ft.; 75 ft.

4. The space passed over in 5 seconds is 105 ft., and the final velocity 35 ft.; find the initial velocity and the acceleration.—*Ans.* 7 ft.; 5·6 ft.

5. A body moving from rest is observed to move over m ft. and n ft. respectively in 2 consecutive seconds; show that the acceleration is $n - m$, and the time from rest $\frac{n}{n-m} - \frac{3}{2}$ ft.

6. A body uniformly accelerated moves at the end of 2 seconds with a velocity which would carry it through 5 miles in the next ½ hour; find the acceleration.—*Ans.* $7\cdot\dot{3}$ ft.

7. A body moving with uniform acceleration describes 574 ft. in the eighth second; find the acceleration.—*Ans.* $76\frac{8}{15}$ ft.

8. A body has described 392 ft. from rest in 7 seconds; find the velocity acquired.—*Ans.* 112 ft.

9. A body has described 54 ft. from rest in 3 seconds; find the time it will take to move over the next 120 ft.—*Ans.* 2·38 secs.

10. A body moves over 70 ft. in the fourth second; find the acceleration.—*Ans.* 20 ft.

11. A body describes 354 ft. while its velocity increases from 43 to 75 feet per second; find the whole time of motion and the acceleration.—*Ans.* 6 secs.; 5·3 ft.

12. A body in passing over 135 ft. has its velocity increased from 7 to 53; find the whole space described from rest and the acceleration.—*Ans.* $137\frac{73}{184}$ ft.; $10\cdot\dot{2}$ ft.

13. Find the numerical value of the acceleration when in a quarter of second a velocity is produced which would carry a body over 2 ft. in every third of a second.—*Ans.* 24.

14. A body moving from rest is observed to pass over 36 ft. and 44 ft. respectively in 2 consecutive seconds; find the acceleration and the time from rest.—*Ans.* 8 ft. 4 and 5 secs.

II.—FORCES PRODUCING MOTION.

18. Hitherto we have considered the motion of a particle under various circumstances independently of the forces which produce the motion. We have now to consider the relations between the forces and the motion which they produce.

The science of Dynamics rests on certain fundamental principles termed the Laws of Motion.

The first law of motion, or the law of inertia, is as follows :—

When a body is not acted on by any force, if it be at rest it will remain at rest, and if it be in motion it will continue to move in a straight line with uniform velocity.

As this law is the basis of Statics as well as Dymamics it was stated at the introduction, together with the evidence on which it rests.

On this and the following law the theory of the motion of the heavenly bodies is based, and the uniform agreement of the deductions from these laws and observations in astronomy is one of the strongest confirmations of their truth.

19. The second law may be stated thus :—

When a force acts upon a body in motion, the change of motion is the same in magnitude and direction as if the force acted on the body at rest.

The truth of this law is shown by such considerations as the following :—

When a person is on board a boat which is moving *uniformly along* a stream, any movement he makes

produces exactly the same effect as if the boat were at rest.

When a stone is let fall from a point on land it falls in the direction of the vertical, and when a stone is let fall from the mast of a ship in motion, it reaches the deck at a point vertically below the starting point. Now the stone falls from the mast to the deck in the same time whether the vessel be at rest or in motion; again, if the vessel passes horizontally through any distance, 3 ft. suppose, during the fall the stone also passes through 3 ft. horizontally, that is, through the same space as it would have passed through had it remained at the top of the mast. We conclude, therefore, that the horizontal motion due to the velocity of the vessel, and the vertical motion due to the attraction of the earth, have each their full effect in their own direction, that is to say, in the resultant motion the stone is displaced horizontally in a certain time exactly as if its vertical motion did not exist, and it is displaced vertically in the same time as if its horizontal motion did not exist.

This will be made still plainer by reference to the following experiments:—

Take a board A B E (Fig. 117) which has a curved groove A B cut in it, so that the direction of the curve at B is horizontal. If a ball *m* be made to slide down this groove the direction of its motion is horizontal. If the groove be prolonged horizontally to the point M, and the surface of the groove and ball be so well polished that friction may be disregarded, after the ball has passed B it will move along B M with uniform velocity. At the instant the ball *m* passes the point B,

let another ball m' fall vertically along B E, and let B M be taken at such a length that m moves from B to M in

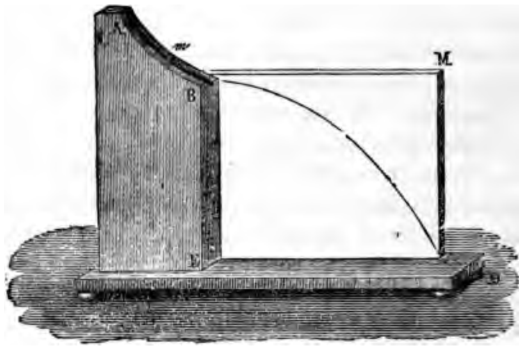


Fig. 117.

exactly the same time as m' moves from B to E. If now the horizontal groove be removed, and the ball be allowed to slide down the groove A B, and then to fall freely, it will neither arrive at the point M nor the point E, but at a point D, having passed through a vertical distance B E and a horizontal distance B M. It will arrive at the point D in exactly the same time as the ball m' falling free would pass from B to E or the ball m after descending through the arc A B would pass from B to M.

Falling Bodies.

20. The fall of bodies to the earth in various circumstances offers remarkable illustrations of the preceding *principles.*

When bodies of different material fall in the air, they do not usually pass through the same heights in the same time. A ball of lead and a scrap of paper fall through the air with very different velocities. The difference arises from the difference in the resistance offered by the air, which varies with the form and dimensions of the body and with the velocity.

If the bodies are made to fall in a tube from which the air has been expelled, then the time of descent and the velocity acquired will be the same. The motion of all bodies in vacuo is uniformly accelerated.

It is usual to call the force producing the motion "gravity," and to indicate the acceleration by g. Hence g is the number of feet added to the velocity of a body moving freely in vacuo for every second of time. This acceleration is not absolutely the same at all points on the earth's surface, but its variations are very small. It increases with the latitude of the place and decreases with the height above the sea. In London a velocity of nearly 32·2 feet is added in every second when a body moves in vacuo.

We may therefore at once apply the formulæ for uniform acceleration to the case of falling bodies. When the body falls from rest the equations of motion (§ 15) are

$$v = gt \qquad s = \tfrac{1}{2} g t^2$$

and by equating values of t from both equations $v^2 = 2 g s$.

When the body has an initial velocity v these equations of motion are

$$v = V \pm gt \qquad S = Vt \pm \tfrac{1}{2} g t$$

and by finding the value of t from the first equation and substituting in the second

$$v^2 = V^2 \pm 2\,gs.$$

In all cases in which the body moves in a vertical line the space passed through is equal to the mean velocity multiplied by the time.

Example 1. A body falling under the action of gravity has a velocity of 30 ft.; how far will it fall in 5 seconds?

Initial velocity	=	30 ft.
Final ,,	=	30 + 5 × 32
Mean ,,	=	110
		5
Space ,,		550 ft.

Example 2. A body is projected vertically upwards with a velocity of 100 feet per second; how far will it ascend in 3 seconds?

Initial velocity	=	100
Final ,,	=	100 — 3 × 32
Mean ,,	=	52
		3
Space ,,		156

Example 3. To find the space passed through in the n^{th} second :—

Velocity at commencement of n^{th} second $= \overline{n-1}\cdot g$.

Velocity at end of n^{th} second $= n\,g$.

Mean velocity and space $= \frac{1}{2}(2\,n - 1)\,g$.

Substituting successively 1, 2, 3, &c. for n, we find the spaces passed through in the successive seconds are $\frac{g}{2}$, $3\,\frac{g}{2}$, $5\,\frac{g}{2}$, $7\,\frac{g}{2}$, &c., and generally the space passed through in the n^{th} second is the n^{th} odd number multiplied by $\frac{g}{2}$.

Fig. 118 will therefore represent the divisions of the path corresponding to different times, each part representing $\frac{g}{2}$ feet.

The student is recommended to consider well the graphic representation of the motion of falling bodies in Figs. 114 and 118, and to keep them constantly in remembrance while solving the following exercises.

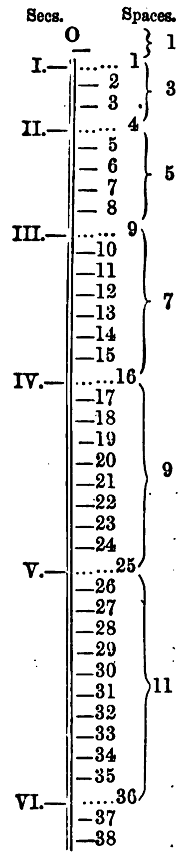


Fig. 118.

Example 4. A body is projected vertically upwards with velocity v; to what height will it rise?

The velocity at any point is given by the equation $v^2 = \mathrm{v}^2 - 2\,g\,s$; but at the highest point the velocity is o. Hence, if h be the height $0 = \mathrm{v}^2 - 2\,g\,h$,

$$\text{hence } h = \frac{\mathrm{V}^2}{2g}$$

The velocity acquired in falling from this height is given by the formula $v^2 = 2\,g\,h$; hence $v = \mathrm{v}$, and the body strikes the ground with a velocity equal to the velocity of projection.

Example 5. Find the time of ascent.

Here $\mathrm{s} = \mathrm{v}\,t - \frac{1}{2}\,g\,t^2$

Put $\mathrm{s} = 0$, then $\mathrm{v}\,t - \frac{1}{2}\,g\,t^2 = 0$,

and $t = 0$, or $\dfrac{2\mathrm{V}}{g}$

The first value corresponds with the instant of starting, and the second with the instant at which the body returns to the starting point.

Again, put $\mathrm{S} = \dfrac{\mathrm{V}^2}{g}$, that is, the height to which the body will rise,

$$\text{then } t = \frac{\mathrm{V}}{g}$$

Now, this is half the whole time; hence the time of ascent is equal to the time of descent.

These results are important, and may be collected thus:—

When a body is projected vertically upwards with velocity v, *it rises for* $\dfrac{\mathrm{V}}{g}$ seconds, *reaches the height*

$\frac{V^2}{2g}$, *falls in the same time as it took to ascend, and strikes the ground with the velocity* v.

21. Let us now apply the second law of motion to find the position of a particle projected horizontally with velocity v, and allowed to fall under the action of its weight, as represented in Fig. 117.

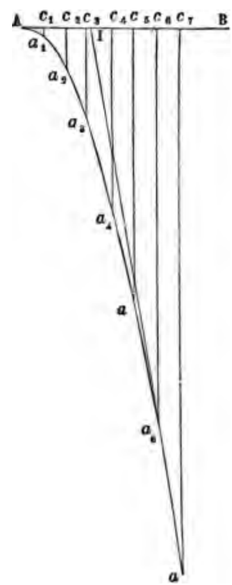


Fig. 119.

Let A B represent the horizontal direction, and, according to a fixed scale, take equal distances A c, $c_1 c_2$, &c. to represent the spaces which would be passed over in successive seconds if the body were subject only to the force of projection. At the points c_1, c_2, &c. draw vertical lines representing, according to the scale chosen, the spaces which would be passed through in 1, 2, 3, &c. seconds if gravity were the only force. Thus, $c_1 a_1 = 16$ ft., $c_2 a_2 = 16 \times 4$ ft., $c_3 a_3 = 16 \times 9$ ft., &c. The points a_1, a_2, a_3, &c. represent the position of the particle at the end of the successive seconds, and the curve joining these points represents the path.

Morin's Apparatus.

22. If the particle could mark out its path on a vertical plane, it would trace out the curve $A\, a_1\, a_2$. If the plane could move horizontally while the body fell vertically, the same curve would be traced out on the plane. This fact is employed in Morin's apparatus, constructed to illustrate the laws of falling bodies.

It consists of a cylinder with vertical axis (Fig. 120), made to revolve uniformly by means of a weight, the regularity of the motion being secured by a fly-wheel at the top. The cylinder is surrounded with paper ruled with horizontal and vertical lines. A cylindrical weight f carries a pencil, the point of which is pressed gently against the surface. The weight is detached by pulling the cord g, and is guided in its fall by two wires. If the cylinder did not revolve the pencil would trace a vertical line upon the surface, and if the cylinder revolved but the pencil remained stationary, a horizontal line would be traced. When, however, the cylinder turns and the weight falls, the curve represented in Fig. 121 is described. The lines $c_1\, a_1$, $c_2\, a_2$ represent the spaces described while the cylinder turns through the arcs $A\, c_1$, $A\, c_2$, &c. When these lines are measured they are found to be as the numbers 1, 4, 9, 16, thus confirming the theory of falling bodies. Moreover, the space passed over in a given time can be measured on the cylinder, and the value of the acceleration due to gravity can then be *calculated.*

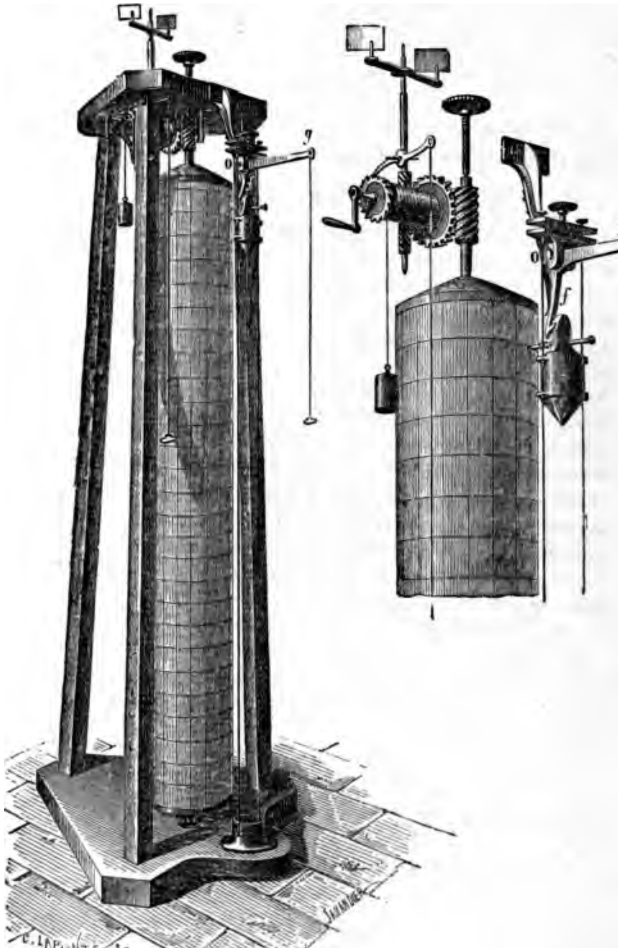


Fig. 120.

EXERCISES.

In the following exercises use 32 ft. as an approximation for the acceleration due to gravity.

On the formula $v = V \pm g t$.

1. A body falls for 8 seconds; with what velocity is it moving at the end of that time?—*Ans.* 256.

2. If a body is let fall, how long will it take to acquire a velocity of 500 ft. per second?—*Ans.* $15\frac{5}{8}$ secs.

3. A body is projected upward with a velocity of 80 ft. per second; determine the velocity it will have at the end of 3 seconds, and the number of seconds that must elapse before its velocity equals its initial velocity.—*Ans.* 16 ft.; 5 secs.

4. A body is thrown downward with a velocity of 160 ft. per second; find its velocity at the end of 5 seconds, and the number of seconds in which a body that is merely dropped would acquire that velocity.—*Ans.* 320; 10 secs.

5. A body A is projected downward with a velocity of 240 ft. per second; at the same instant another body B is projected upward with an equal velocity; how much faster will A be moving than B at the end of 6 seconds?—*Ans.* 9 times.

6. A body is thrown upward with a velocity of 96 ft. per second; with what velocity will it be moving at the end of 4 seconds?—*Ans.* Downwards with a velocity of 32 ft. per second.

7. A body is dropped from a certain height h, at the same instant as another is thrown upward; what initial velocity must the latter have that it may meet the former half way?—*Ans.* $\sqrt{gh}$.

8. A body is at a given instant moving upward with a given velocity v; show that it will be moving downward with an equal velocity after $\frac{2v}{g}$ seconds, and that it will reach its highest point after $\frac{v}{g}$ seconds.

9. A body is thrown up with a velocity $a\ g$; after how long will it be descending with a velocity $b\ g$?—*Ans.* $a + b$.

On the formula $S = Vt \pm \frac{1}{2} g t^2$

10. How many feet will be described in 7 seconds by a body that moves freely from rest under the action of gravity?—*Ans.* 784.

11. Through how many yards would a body falling freely from rest descend in 3 minutes?—*Ans.* 172,800.

12. A body is projected downward with a velocity of 50 ft. per second; how far will it fall in $4\frac{1}{2}$ seconds?—*Ans.* 549 ft.

13. A body is projected upward with a velocity of 150 ft. per second; how high will it have ascended in $6\frac{1}{2}$ seconds?—*Ans.* 299 ft.

14. If a body is thrown upward with a velocity of 96 ft. per second, how far will it be from the starting point at the end of $4\frac{3}{4}$ seconds, and what will be the whole space it will have described?—*Ans.* 95 ft.; 193 ft.

15. A body is projected upward with a velocity of $3g$ ft. per second; determine the height of the body, and with what velocity, and in what direction, it will be moving at the end of 4 seconds.—*Ans.* Height $4g$, moving downward with velocity g.

16. A body is projected upward with a velocity v; show that it will return to the point of projection after $\frac{2v}{g}$ seconds.

17. A body falls for a time t, and has a velocity V at the beginning, and v at the end of that time; find the space described. —*Ans.* $\frac{1}{2}(V + v)\ t$.

On the formula $v^2 = V^2 \pm 2\,g\,s$.

18. If a body is thrown upward with a velocity of 36 ft. per second, find its greatest height.—*Ans.* $20\frac{1}{4}$ ft.

19. If a body falls freely through 1,600 ft., find the velocity it acquires.—*Ans.* 320 ft.

20. A body is projected vertically upward with a velocity of 100 ft. per second; how long will it take to reach the top of a tower 100 ft. high?—*Ans.* $1\frac{1}{4}$ or 5 secs.

21. If a body is thrown upward with a velocity v, show that it will ascend through $\frac{v^2}{2g}$ ft.

22. Two particles are let fall from two given heights; find the interval between their starting if they reach the ground at the same time.—*Ans.* $\frac{\sqrt{2}}{\sqrt{g}}(\sqrt{h} - \sqrt{h'})$.

23. A stone A is let fall from a certain point, and after it has fallen for a second, another stone B is let fall from a point 100 ft. lower down; in how many seconds will A overtake B?—*Ans.* $3\frac{1}{8}$ seconds.

24. A stone is dropped into a well and is heard to strike the water in 3 seconds; if the velocity of sound $= 35\,g$, find the depth of the surface of the water.

(Let $x =$ the depth, show that the time of fall of the stone $= \frac{\sqrt{2x}}{\sqrt{g}}$ and time of passage of the sound $= \frac{x}{35\,g}$; thus $\frac{x}{35\,g} + \sqrt{\frac{2x}{g}} = 3$, therefore $x = 4{\cdot}155\,g$.)

25. A stone A is projected vertically upward with a velocity of 80 feet per second; after 4 seconds, another stone B is let fall from the same point; how long will B move before it is overtaken by A, and how far will they then be from the point of projection?—*Ans.* $1{\cdot}3$; $28{\cdot}\dot{4}$.

26. In the last example, if only 2 seconds had elapsed, would A ever have overtaken B?

27. The point A is $4\,g$ ft. above B; a body is thrown upward from A with a velocity of $2\,g$ ft. per second, and at the same instant another is thrown upward from B with a velocity of $3\,g$ ft. per second; show that after 4 seconds they will both be at A, moving downward with velocities $2\,g$ and g feet respectively.

28. A body was thrown from a height $8\,g$ and struck the ground with a velocity $5\,g$; what was the velocity of projection?—*Ans.* $3\,g$.

III.—PROJECTILES.

23. We will now discuss the properties of the path of a projectile at greater length, and shall have occasion to refer to the following facts and definitions.

If a fixed straight line and a fixed point be taken, the curve containing all points which are at the same distance from the line as from the point, is termed a parabola. The fixed straight line is termed the directrix of the parabola, and the fixed point the focus. A straight line through the focus perpendicular to the directrix is termed the axis, and the point in which the axis cuts the curve is termed the vertex.

From the definition of a parabola it is proved in Geometry that if a tangent, O C, be drawn at any point O in a parabola, and then from another point B a straight line B C be drawn parallel to the axis, the ratio $\frac{(OC)^2}{BC}$ is constant, and equal to four times the distance of the point O from both the focus and the directrix of the parabola.

In the theory of projectiles we omit the consideration of the resistance of the air, and suppose the body to move in a perfect vacuum.

The path of a Projectile is a Parabola.

24. It is evident that the path will be in the vertical plane through the line of projection: let it lie in the plane of the paper, and let O be the point of projection, *O M the line* in which the body is projected; O M will *manifestly* be a tangent to the curve described.

Let V be the velocity of projection, and C the point at which the body would arrive with this velocity in the time t, so that $OC = Vt$ (Fig. 121).

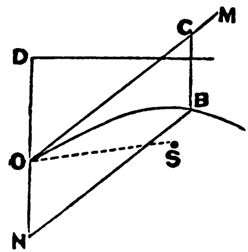


Fig. 121.

From C draw CB vertical, and make $CB = \frac{g\,t^2}{2}$; then, if there had been no velocity of projection, CB would be the space which the body would describe in the time t under the action of gravity only; and if gravity had not acted, the body would have been at C; therefore when the body is simultaneously animated by its original velocity V, and that generated by gravity, it will be at B.

Complete the parallelogram ONBC, then

$$ON = CB = \frac{g\,t^2}{2}$$

$$BN = AC = V\,t$$

$$\therefore \; BN^2 = V^2 t^2 = \frac{2\,V^2}{g} \cdot ON$$

Hence $$\frac{BN^2}{ON} = \frac{2V^2}{g}$$ = a constant, and therefore B lies in a parabola, of which the axis is vertical,

and the distance of M from the directrix or focus is $\frac{V^2}{2g}$.

25. From the point O draw OD perpendicular to the directrix and meeting it in D, and let S be the focus, then $V^2 = 2g \cdot OD = 2g \cdot OS$.

Hence, the velocity at any point of the parabola is that which would be acquired in falling from the directrix. The distance of the directrix is therefore independent of the angle of projection. If, then, a number of particles were projected from the point O with the same velocity at the same instant, in different directions, one of which is vertical, all the parabolas described would have a common directrix which would be just reached by the particle whose path is vertical.

To determine the greatest height to which the Projectile will rise.

26. The velocity of projection may be supposed to consist of two velocities, one horizontally $= V \cos a$, the other vertical $= V \sin a$. It is evident that the horizontal velocity will remain the same, for every point on the curve for gravity acts vertically, and can produce no effect horizontally. The vertical motion of the body will be the same as if it were projected vertically with the velocity $V \sin a$; hence, if h be the height required (OE Fig. 122) we have (§ 20)

$$h = \frac{V^2 \sin {}^2 a}{2g}$$

To find the distance of the Focus from the vertex of the Parabola.

27. At the highest point the vertical velocity is o, and the horizontal velocity v cos α. We may suppose the body projected at any point with the velocity and direction it has at that point. Suppose the body projected at the point A (Fig. 119), then the equation of Fig. 121 becomes

$$\frac{(AC)^2}{CA} = \frac{2(V \cos \alpha)^2}{g}$$

hence the distance of the focus from the vertex equals $\frac{V^2}{2g} \cos^2 a$. Four times this distance is termed the latus rectum.

To find the range of the Projectile, that is, the distance from the point of projection at which the body meets the horizontal plane through the point of projection.

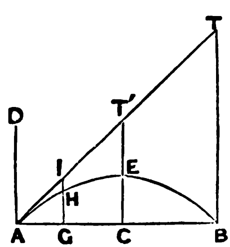


Fig. 122.

28. The range will evidently be twice the distance of the point of projection from the axis of the parabola. Let A be the point of projection, A T the direction of projection, A B the range on the horizontal plane through B, and T B vertical. Let v be the velocity *of projection, and* the angle T A B, and T the time of

flight, that is, the time the particle takes to pass from A to B, then

$$AT = Vt, \qquad TB = \tfrac{1}{2} g t^2; \text{ hence}$$

$$\sin a = \frac{TB}{AT} = \frac{\frac{1}{2} g t^2}{V t} = \frac{g t}{2V}$$

$$\therefore \text{ Time of flight } = \frac{2V \sin a}{g}$$

$$\text{And } AB = AT \cos a = V \cdot t \cos a$$

$$= \frac{2V^2 \sin a \cos a}{g}$$

$$\text{But } 2 \sin a \cos a = \sin 2a$$

$$\therefore AB = \frac{V^2}{g} \sin 2a$$

The greatest value which $\sin 2a$ can have is 1, which corresponds to $a = 45°$; hence the range of a projectile will be greatest when the angle of projection is 45°.

To find the Equation to the Curve referred to Rectangular Co-ordinates.

29. We find the equation when we express the relation between the variable lengths AG, GH, for any point H of the curve (Fig. 122).

Let $AG = x$, $GH = y$, $GAI = a$, and $t =$ the time of describing AH.

Then $AI = vt$ and $x = AI \cos a = vt \cos a$.

Also $GI = x \tan a$; and $HI = \frac{1}{2} g t^2$;

hence $y = GI - HI = x \tan a - \frac{1}{2} g t^2$;

but from the first equation $t = \frac{x}{V \cos \alpha}$

hence the equation is $y = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha}$.

These Equations

$$\text{I. } x = V t \cos \alpha$$

$$\text{II. } y = x \tan \alpha - \tfrac{1}{2} gt^2$$

$$\text{III. } y = x \tan \alpha - \frac{gx^2}{2 V^2 \cos^2 \alpha}$$

are frequently useful in solving problems. For example:—*To find the range on an inclined plane.* Let θ be the inclination of the plane and l the range, then in Equation III. substitute $x = l \cos \theta$, $y = l \sin \theta$.

To find the time of flight on an inclined plane. Find l the range and substitute $x = l \cos \theta$ in I.

To find the angle of projection so as to hit a given object. The height and distance of the object must be known, and hence they may be substituted for y and x in Equation III., then remembering that $\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha$, we shall have a quadratic to determine $\tan \alpha$.

30. This theory of projectiles rests on the supposition that the motion takes place in a non-resisting medium, but the resistance of the air affects the motion of heavy bodies so materially as to render the parabolic theory nearly useless in practice. The path inclines to the earth more rapidly than is the case with a parabola: hence the range is much less. For example, when the velocity is about 2,000 ft., the resistance of the air is 100 times the weight of the ball and the greatest range, which, according to theory should be 23 miles, is less than 1 mile.

The calculation of the exact modification of the path caused by this resistance must, however, be postponed for the present; it will suffice here to remark that it admits of precise estimation. With guns and rifles sighted by theory great precision of aim has been attained.

The sight of a rifle (Fig. 123) is usually graduated,

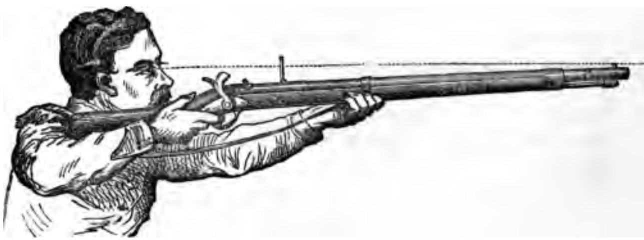


Fig. 123.

so that the angle of projection (the angle between the line of sight and the barrel) may be arranged according to the distance of the object. With long ranges the height of the path is considerable; the trajectory of the Enfield rifle at a range of 1,000 yards is, at its highest point, 70 ft. above the horizontal plane.

Exercises.

1. A body is projected at an angle a and with velocity V; find the vertical velocity at the time t.—*Ans.* $V \sin a - g t$.

2. A body is projected with velocity of 60 ft. per second in a direction making an angle of 15° with the horizon; find its velocity at the end of half a second.—*Ans.* 57·95. (Find the re-

sultant of the horizontal and vertical velocities.—(g = 32·2, sin 15° = ·2588).)

3. In the above example find the range, greatest altitude, and time of flight.—*Ans.* 55·916 ft.; 3·745 ft.; ·965 sec.

4. If the angle of projection be 45° and greatest height 125 ft., find the velocity of projection, the range, and time of flight.—*Ans.* 126·86; 500; 5·57 secs.

5. The velocity of projection is 259·21 ft.; find the distance of the focus of the parabola from the point of projection.—*Ans.* 1043·32025.

6. The velocity of projection is $40\frac{1}{4}$ ft.; find the height of the directrix of the parabola.—*Ans.* $25\frac{5}{32}$ ft.

7. The horizontal range = 1,000 ft., the time of flight = 15 secs.; find the direction and velocity of projection, and the greatest height.—*Ans.* 74° 33′ 48″ (sin = ·963925); 250·5 ft.; 905·36 ft.

8. Velocities of 35 ft. and 12 ft. per second, in directions at right angles to each other, are simultaneously communicated to a body; determine the resultant velocity.—*Ans.* 37.

9. If at the highest point of the path of a projectile the velocity be altered without altering the direction of motion, will the time of reaching the horizontal plane which passes through the point of projection be altered?

10. Show that the greatest range up a plane inclined at 30° is two-thirds of the greatest range on a horizontal plane, the initial velocity being the same in the two cases.

11. A body is projected with the velocity $5g$ at an inclination of 75° to the horizon; determine the range.—*Ans.* $12\frac{1}{2}g$.

12. If two projectiles have the same initial velocity and the same horizontal range, the foci of their paths are at equal distances from the horizontal plane.

13. A body is projected with a velocity V and angle of projection α; determine the velocity with which another must be projected vertically, so that the two may reach the ground at the same instant.—*Ans.* V sin α.

14. A heavy particle is projected from a point with a velocity of 90 ft. and in a direction inclined 30° to the horizon; find its

distance from the point of projection at the end of 2 seconds. —*Ans.* 157·97 ft.

15. A number of particles are projected from a fixed point in one plane, so that their least velocity is constant; show that all of them will be found at the same instant on the same vertical line.

16. The horizontal range of a projectile is three times the greatest altitude; find the angle of projection.—*Ans.* $53^\circ\ 8'$ (tan $= \frac{4}{3}$).

17. A body is projected with velocity of 150 ft. per sec. and angle of projection 60°; find the direction, velocity, and height at the end of 5 seconds.—*Ans.* $V = 81{\cdot}177$ ft.; $H = 247{\cdot}13$ ft.; $\alpha = 157^\circ\ 30'$ ($\sin \alpha = {\cdot}382523$).

18. The height of a projectile at a distance of 880 yards is 100 ft., and the whole range 1,200 yards; find the velocity and direction of projection.—*Ans.* 645·1 ft.; $8^\circ\ 5'$ ($\tan = \frac{25}{176}$).

19. Two bodies are projected with the same velocity V and making equal angles (α) with the vertical from a point on an inclined plane ($\angle\ \theta$), one moving directly up the plane and the other down; compare the ranges.—*Ans.* $\frac{\cos(\alpha + \theta)}{\cos(\alpha - \theta)}$.

20. Show in the above case that the ratio of the times of flight is equal to that of the ranges.

IV.—MASS.

31. Hitherto we have considered a force producing motion as measured by the acceleration or the velocity it generates. It is, however, necessary to take into account the quantity of matter moved. When the same *force acts on different* bodies, it produces in each a

uniform acceleration; but the acceleration will not be the same in both cases.

This difference arises from what is called the mass of the bodies.

32. *Mass* is a term for the quantity of matter in a body. It is not easy to give such a definition of mass as will lead us at once to a method of measuring it. We are compelled to measure mass by its effects.

We assume that the attraction of the earth on all particles of matter is the same, and is not dependent on the nature of the matter attracted. This assumption seems to be justified by the fact that bodies of all kinds fall with equal velocity in the exhausted receiver of an air-pump. Hence we measure the mass of a body by its weight, and can only define the mass as a quantity proportional to the weight. If, then, at the same spot in the earth's surface one body is twice as heavy as another, the mass of the first is twice that of the second.

Suppose, however, the body is weighed by a spring balance (Fig. 2) at a certain place, and weighed again by the same instrument at another place nearer the equator, it will be found that the body is lighter at the latter place. It is found also that the acceleration due to the attraction of the earth is also less at the second place than at the first in the same proportion. This illustrates the fact that when the mass remains the same, the weight varies as the acceleration due to gravity. Consequently, writing M for mass, W for weight, and g for acceleration due to gravity,

W varies as M when g is constant,

W varies as g when M is constant;

therefore W varies as Mg when both vary; and therefore, by a theorem of algebra, if

$$C = \text{some constant quantity,}$$
$$W = CMg.$$

By selecting a suitable unit of mass we may make this constant unity;

$$\text{then } W = Mg.$$

This formula exhibits a relation between the statical and the dynamical measure of force. When a force acts upon a particle, the dynamical effect of the force (supposing the particle to move) is measured by the acceleration produced. The statical effect, supposing the particle to be kept at rest, is the pressure produced; and if P be the pressure, M the mass, and f the acceleration, then $P = M \cdot f$.

If for M we write its value in the above equation, we have

$$f = g \frac{P}{W}$$

This result may be expressed in words as follows:—

33. *When pressure produces motion, the acceleration varies directly as the pressure, and inversely as the mass.*

The product Mf is sometimes called the *moving force*, and this statement is then made thus—the moving force is proportional to the pressure. This law may be illustrated by the following arrangement:—

Suppose two weights P and Q, whose masses are respectively m and m', to be connected by a fine inextensible cord passing over a fixed pulley. If $P = Q$ there will be no motion; if P be greater than Q there

will be a pressure producing motion equal to the excess of P over Q or (P — Q). The mass moved is $m + m'$, hence

$$f(m + m') = P - Q,$$

$$\text{but } m = \frac{P}{g} \text{ and } m' = \frac{Q}{g}$$

$$\text{therefore } f = g\,\frac{P - Q}{P + Q}$$

34. *To find the tension in the cord.*

We may consider the tension T to be the upward force acting both on P and Q; the pressure producing the motion of P downwards will be P — T, and the pressure moving Q upwards T — Q, and the accelerations are the same;

$$\text{hence } f = \frac{P - T}{m}$$

$$\text{and } f = \frac{T - Q}{m'};$$

$$\text{therefore } \frac{P - T}{m} = \frac{T - Q}{m'}$$

or dividing by g,

$$\frac{P - T}{P} = \frac{T - Q}{Q}$$

$$\text{consequently } T = \frac{2PQ}{P + Q}$$

35. For example, suppose two weights connected by a cord passing over a fixed pulley to be 3 ozs. and 5 ozs. respectively; then, $f = 32 \cdot \frac{2}{8} = 8$, and if the weights are allowed to move from rest, the equations of motion become $v = 8t$ and $s = 4t^2$. The tension in the cord will be $\dfrac{2 \times 3 \times 5}{3 + 5} = 3\frac{3}{4}$. If the 8 lbs.,

instead of being divided into unequal parts 3 and 5, were divided into equal parts, the tension of the string would be 4 lbs.; hence we see that the tension is less when the parts are unequal than when they are equal. That which is true in this particular case is generally true.

Attwood's Machine.

36. The machine represented in the figure was devised by Attwood for testing experimentally the laws of motion and the results derived from the theory of falling bodies.

It consists of an upright beam, supporting at the upper extremity a nicely constructed wheel turning on a horizontal axis, and two equal weights P, P′, connected by a fine silk thread which passes over a groove in the wheel. To diminish friction the axis of the larger wheel turns on friction wheels (Fig. 124).

The pillar is furnished with a graduated scale, a movable ring *n*, and stage *m*.

Another important appendage to the machine is a small clock with a pendulum beating seconds.

Suppose the weights P and P′ to be at first equal: they will then be at rest.

Let the movable ring be in the position *n*, below the weight P and on this weight place a small bar *p*. P will descend with accelerated velocity until it reaches the ring (*n*). Here the bar *p* will be lifted off the weight P by means of the ring (*n*), and P will move onward with a uniform velocity equal to the velocity at the moment when (*p*) was retained by the ring. Now *place the stage* in such a position that the weight P

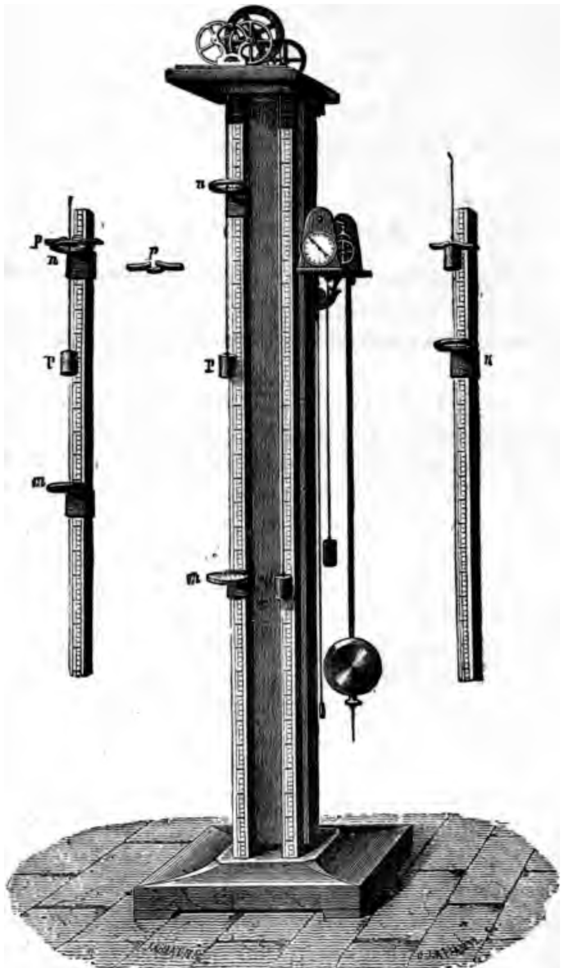


Fig. 124.

may reach it in exactly one second after it has passed the ring (n). The distance between the stage and ring will be the space described by a body moving with a uniform velocity in a unit of time.

Let us now suppose that the ring is exactly so far below the starting-point P as to lift off the bar when P has moved for exactly one second. Also let the stage be so placed as to stop P exactly one second after the bar has been removed. Then the distance $n\,m$ is the acceleration. Now the following relation is found to exist between this acceleration and the weights: if we double the weight of the bar p, while we keep the sum of the weights P P′ and p the same, we double the acceleration, and if we take a bar n times the weight of the first, the second acceleration will be n times that of the first.

If now we keep the bar the same, but double the sum of the weights, we shall find that the acceleration is then only half what it was before, and generally if we multiply the weight moved by n we shall divide the acceleration by n.

Thus the acceleration varies directly as the pressure producing motion, and inversely as the mass moved. In every respect we find the results agree with the theory of Article 33.

The chief advantage secured by this machine is that we may make the acceleration as small as we please, and thus render the motion slow enough to be observed without difficulty.

Suppose, for example, that the equal weights are each 31·7 grammes and the bar 1 gramme, then, taking *the acceleration* due to gravity as 32·2, we have, by the

theory of Article 33, an acceleration equal to $32{\cdot}2 \times \frac{1}{32{\cdot}7 + 31{\cdot}7} = \frac{1}{2}$. Now if we keep the bar on throughout the time it is found that the space described in the first second is $\frac{1}{4}$ ft., in the next second $3 \times \frac{1}{4}$ ft. in the third $5 \times \frac{1}{4}$ ft. and in the nth second the nth odd number multiplied by $\frac{1}{4}$ ft.

Again, by cutting off the pressure producing motion at the end of the successive seconds and measuring the spaces passed through in the next second, it is found that the velocity at the end of the first second is $\frac{1}{2}$ ft., at the end of the next $2 \times \frac{1}{2}$ ft., and at the end of the third $3 \times \frac{1}{2}$ ft., and at the end of the nth second $n \times \frac{1}{2}$ ft. Thus the results of the experiment agree entirely with those of the theory of falling bodies.

Motion down an Inclined Plane.

37. Suppose a body of weight W to rest on a smooth inclined plane; in Statics we have shown that a force P acting in the direction of the plane will support the body if $P = W\frac{h}{l}$, h being the height of the plane, and l the length. Hence $W\frac{h}{l}$ is the resultant of the weight and reaction of the plane, and is the pressure producing motion when the body is free to move down the plane.

Consequently, if f = the acceleration down the plane

$$f = g\,\frac{P}{W};$$

hence, substituting for P

$$f = g\ \frac{h}{l}$$

Thus all the circumstances of the motion of a body down an inclined plane may be found in precisely the same way as in the case of a body falling freely by using $g\ \frac{h}{l}$ as the acceleration instead of g.

To find the velocity acquired by the body in falling down the length of the plane.

38. The formula $v^2 = 2\ f s$ becomes in this case

$$v^2 = 2 \cdot g\ \frac{h}{l} \cdot l = 2\ g\ h$$

But this is the equation which gives the velocity of a body falling freely down a height $= h$.

Hence the velocity acquired in sliding down a smooth inclined plane is the same as would be acquired in falling freely through a vertical space equal to the height of the plane.

The time of falling from rest down a chord of a vertical circle drawn from the highest point is constant.

39. Let A be the highest point of a vertical circle (Fig. 125), A B a diameter, A C any chord.

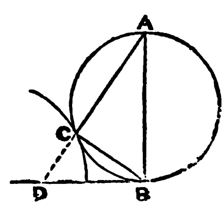


Fig. 125.

Draw the tangent at B meeting A C produced in D. Then the triangles A C B, A B D having an angle A common and the right angles at C and B equal, are similar.

Let t be the time of falling down A C; then

$$\text{A C} = \tfrac{1}{2} t^2 g \frac{\text{AB}}{\text{AD}}$$

$$\text{And } \frac{\text{AC}}{\text{AB}} = \frac{\text{AB}}{\text{AD}} \text{; so that AB} = \tfrac{1}{2} t^2 g.$$

But if t' be the time of falling down A B; then

$$\text{AB} = \tfrac{1}{2} g t'^2$$

That is, t is equal to the time of falling freely down the vertical diameter A B. This establishes the proposition.

In the same manner we may show that the time of falling from rest down a chord passing through the lowest point is constant.

40. This proposition frequently enables us to find what are termed *lines of quickest descent* from a point to a curve or from one line to another. For example, suppose a curve touches the circle in the point C, then A C is the line of quickest descent from A to the curve. If any other point in the curve be joined with A, the line formed will consist of a chord of the circle (the time down which is equal to the time down A C) and a part without the circle. Hence the time down the whole is greater than the time down A C.

For the solution of such problems we must therefore devise a construction which will furnish a circle having the given point for its highest point, and touching the given line. The following constructions should be followed out and demonstrated :—

1. Find the straight line of quickest descent from a point A *to a given line* C B, *neither horizontal nor vertical.*

Draw a horizontal line through A, meeting C B in C,

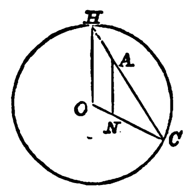
and describe a circle touching A C in A and C B in a point D; join A D, and it will be the line required.

2. From a point A *without a given circle to the circle.*

Join A with the lowest point L of the circle; then if A L cut the circle in D, A D is the straight line of quickest descent.

3. From a point A *within a given circle to the circle.*

Join the point A with the highest point H of the circle, and produce H A to meet the circle again in C; A C is the line required.



Join O H, O C, and draw A N parallel to O H, then ∠ N A C = ∠ O H C. Therefore ∠ N A C = ∠ N C A, hence N A = N C, and a circle having N for centre and N A for radius will pass through A and C, having A for its highest point and touching the first circle in the point C.

4. From a straight line B C *to a circumference below it.*

Draw a tangent L C at the lowest point L of the circle, meeting B C in C; on B C take C A = C L, and draw A L cutting the circle in D; A D is the line required.

It is evident from Example 2 that the line produced passes through L, and it is also evident from Example 3 that it also passes through the point of contact A of

a circle touching the line B C in a point A and the given circle in L.

5. *From one given circle to another.*

Draw a line from H, the highest point of the first, to L, the lowest point of the second, cutting the circles in A and D respectively; A D is the line required.

It is evident from Example 4 that the straight line must pass through H, and from § 39 that it must cut the second circle in the point D, which is the point of contact of a circle drawn to touch the first circle H, and also to touch the second circle.

EXERCISES.

1. How long must a force of ½ lb. act on a weight of 100 lbs. to generate a velocity of 3·g ft. per second?—*Ans.* 10 minutes.

2. The velocity of a mass of 100 lbs. under the action of a force increases from 30 to 80 ft. per second in 5 seconds; find the moving force.—*Ans.* $\frac{1000}{g}$ lbs.

3. Two equal weights, each 18·6, are suspended over a fixed pulley; what must be added to one of them that it may descend through 100 ft. in 8 seconds ($g = 32{\cdot}2$)?—*Ans.* 4 oz.

4. A mass of 18 lbs. is divided into two parts, which are suspended from a cord passing over a fixed pulley; the heavier part descends through 13 yards in 13 seconds; find the weights.—*Ans.* 9·129 lbs.; 8·871 lbs.

5. A force of ½ oz. draws a weight of 80½ oz. along a smooth horizontal plane; what will be the space passed through in 5 seconds?—*Ans.* 2·5 ft.

6. A weight of ½ oz. hanging vertically from a cord passing over the edge of a smooth table draws a weight of 49½ oz. along the table; what will be the space passed through in 5 seconds?—*Ans.* $\frac{g}{8}$ ft.

7. *Compare this question with the preceding.*

8. *If in Fig.* 124 P and P′ weigh 5 lbs. each, and p weighs

half a pound, find the space described in 7 seconds, supposing that when p is laid on P an initial velocity of 3 ft. per second is communicated to it.—*Ans.* $58 \cdot 5\dot{6}$ ft.

9. A body falls 9 ft. along an inclined plane in the first second; find the inclination of the plane.—*Ans.* Ht. : L. as 9 : 16·1, or the inclination = 34°.

10. The length of an inclined plane is 400 ft., its height 250; a body falls from rest from the top of the plane; what space will it have fallen through in $3\frac{1}{2}$ seconds; what time will it be in falling through 300 ft.; and what velocity will it have acquired when it has arrived within 50 ft. of the bottom of the plane?—*Ans.* 123·23 ft.; 5·46 seconds; 118·67 ft.

11. A body slides down a smooth inclined plane of given height; show that the time of its descent varies directly as the length of the plane.

12. Find the position of a point on the circumference of a circle, so that the time of descent down an inclined plane to the centre of the circle may be equal to the time of descent down an inclined plane to the lowest point of the circle.—*Ans.* 60° from the highest point.

13. If f be the measure of an acceleration when m seconds is the unit of time, and n ft. the unit of length, find the measure of acceleration when a second and a foot are the units.—*Ans.* $f \cdot \frac{n}{m^2}$.

14. Two weights connected by a string, passing over a fixed pulley, move for 3 seconds and are then at the same level; if the string be suddenly cut, find the distance between them in 3 seconds more.

15. A smooth plane is inclined at an angle of 30° to the horizon; a body is started up the plane with a velocity g; find the time it takes to describe a space g.

16. A body falls down a given inclined plane, of length 24 ft., and at the instant when it begins to fall, another is projected upward from the bottom of the plane with a velocity equal to that acquired in falling down an equally inclined plane three times its length; where will they meet?—*Ans.* 2 ft. down.

17. The length of an inclined plane is 40 ft., and its inclination is 30°; mark out upon it a part, equal to the height, through which a body, falling down the plane, will move whilst another body would descend freely through the height.—*Ans.* It must begin $2\frac{1}{2}$ ft. from the top.

18. Divide the length of a given inclined plane into three parts, so that the times of descent down them successively may be equal. (See Fig. 118).—*Ans.* $\frac{1}{9}$, $\frac{3}{9}$ and $\frac{5}{9}$.

19. Find the straight line of quickest descent.

a. From a circle to a straight line below it.

b. From a circle to a point below it.

c. From a given circle to another circle within it.

d. From a given straight line to a point.

20. Show that the velocity acquired in falling from rest down any curve is the same as that acquired in falling down the vertical height of the curve.

(Suppose the curve made up of small straight lines.)

21. A weight of 7 lbs. falls down a rough inclined plane whose height is to the length as 8 : 17, the coefficient of friction being one-third; how far will it descend in 4 seconds?—*Ans.* $\frac{24}{17}g$.

(The pressure producing motion is $\left(\mathrm{W}\cdot\frac{h}{l} - \frac{\mathrm{W}}{3}\frac{b}{l}\right)$

22. If a body be projected down a plane inclined at 30° to the horizon, with a velocity equal to three-fourths of that due to the height of the plane, the time down the plane from rest will equal the time down its vertical height.

23. A body descending vertically draws an equal body 25 ft. in $2\frac{1}{2}$ seconds up a plane inclined at 30° to the horizon, by means of a string passing over a pulley at the top of the plane; determine from these data the force of gravity.—*Ans.* 32.

24. Two bodies start from the top of an inclined plane, one falling down the length of the plane, and the other down its height; it is observed that the former is four times as long as the latter in reaching the base. Required the inclination of the plane.—*Ans.* $\frac{h}{l} = \frac{1}{4}$

25. What velocity of projection must be given to a body thrown up an inclined plane of height $4\frac{1}{2}\,g$, that it may just reach the top, and in what time will it return to the starting point?—*Ans.* $V = 3\,g$; $t = 6$ secs.

26. A given weight P draws another given weight W up an inclined plane of given height and length, by means of a string parallel to the plane; when and where must P cease to act, that W may just reach the top?—

Ans. $S = \frac{P + W}{P} \cdot \frac{hl}{l + h}$; $t = \frac{P + W}{P} \sqrt{\frac{2Phl^2}{(Pl - Wh)\,(h + l)g}}$

V.—CENTRIFUGAL FORCE.

41. When a body describes a circle of radius r, with uniform velocity v, we have seen that the normal acceleration is $\frac{v^2}{r}$ (§ 17).

Consequently, in order to keep the body moving in a circle there must be a force acting upon it which will produce a constant acceleration towards the centre, equal to $\frac{v^2}{r}$. Hence, if W be the weight of the body, P the pressure tending towards the centre, then

$$\frac{v^2}{r} = \frac{P}{M} = g\,\frac{P}{W}.$$

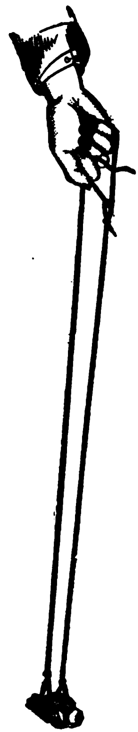


Fig. 126.

A pressure equal and opposite to P is sometimes spoken of as the centrifugal force.

In illustration of this formula, let a stone be tied to the end of a string and whirled round (Fig. 126). Suppose the string to be r feet long, the stone to weigh W lbs., and to move with a velocity of v feet per second; now, the tendency of the stone at each instant is to move off in the direction of a tangent to the circle it describes (§ 18), therefore there must be exerted on it at each instant a certain pressure P acting along the radius and towards the centre sufficient to deflect it from the tangent and to keep it in the circle; this pressure is given by the formula

$$P = \frac{W}{g} \cdot \frac{v^2}{r}$$

In the case supposed, the pressure is supplied by the hand, and gives rise to the same sensation as would be produced if the stone were pulled outward with a pressure of P lbs. Whenever a heavy body moves in a circle under the action of any forces whatever, the sum of the resolved parts of the forces along the radius must at each instant equal $\frac{W}{g} \cdot \frac{v^2}{r}$, or the body *will not continue* to move in the circle.

Example. Suppose the string in Fig. 126 capable of sustaining a weight W at rest equal to six times the weight of the stone; if the stone describes a circle 6 ft. in diameter, required the number of revolutions per minute when the string is on the point of breaking.

The normal acceleration is $\frac{v^2}{r}$; hence, if T is the tension,

$$\frac{v^2}{r} = g\frac{T}{W}.$$

When, however, the string is on the point of breaking

$$T = 6W;$$

$$\text{hence} \quad \frac{v^2}{3} = 32 \times 6;$$

$$\text{and } v = 24.$$

Now the circumference is 6π, therefore the stone which describes 24 ft. in 1 sec. describes 6π in $(6\pi \div 24)$ secs. and makes $60\,(24 \div 6\pi)$ per minute.

Another example of centrifugal force will be afforded by a carriage describing a curve of small radius. Let G be the C. G. of the vehicle (Fig. 127), then, whether the carriage be at rest or in motion along a straight line, the weight will act only in the direction of the vertical G A which falls within the wheels, and consequently the equilibrium is stable.

Suppose now the carriage to describe a curve of small radius, then a horizontal centrifugal force will be generated.

Let G A represent the weight, and let G B, according to the same scale, represent the centrifugal force; these forces will have a resultant G C. In the figure, G C falls

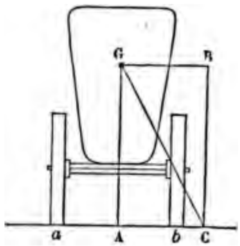


Fig. 127.

without the wheels, and, consequently, the carriage will be overturned. The tendency to overturn may be counteracted by inclining the plane on which the body moves. On railroads, where sharp curves occur, the outer rail is made higher than the inner, so that the reaction of the rails is not vertical but has the direction of the resultant of the centrifugal force and the weight of the train.

42. When a body turns about a fixed axis, each point in it describes a circumference in a plane perpendicular to the axis of rotation, having a radius equal to the distance of the point from the axis (Fig. 128).

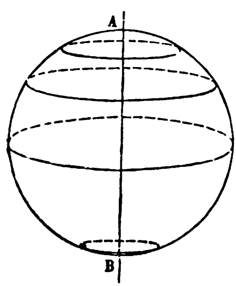


Fig. 128.

If the rotatory motion is uniform, every point describes an arc, whose length is proportional to the time, and its velocity is thus measured by the length of the arc described in one second.

As all the points of the same body describe an entire circumference in the same time, and as the circumferences of all circles are proportional to their radii, it is evident that the velocities of the points of the same *rotating* body change at different points, and are *inversely* proportional to the distance from the axis.

Thus, at the surface of the earth the velocity is greatest at the equator, and diminishes as we advance towards the poles, so that at each pole it is nothing, since these points turn on themselves without displacement.

If the earth were at rest it would be a sphere differing only from a perfect sphere by the inequalities of its surface, which are exceedingly small compared with its size. The motion round its axis causes the land at the equator to bulge out, whilst the poles are flattened.

This effect may be illustrated by means of the apparatus represented by Fig. 129.

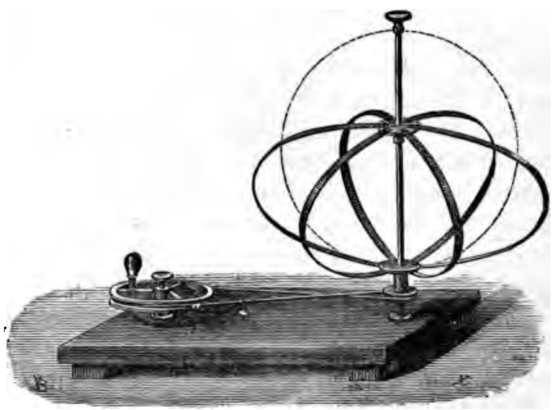


Fig. 129.

The instrument consists of two circles turned about a vertical axis by a multiplying wheel turned by the hand.

In proportion as the velocity of rotation increases, the diameter extends in a direction perpendicular to the axis, and diminishes in the direction of the axis.

43. To the variation of the centrifugal force and to the spheroidal shape of the earth is due the variation in the force of gravity.

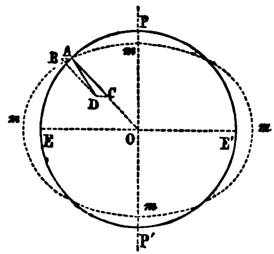


Fig. 130.

Let P P′ represent the axis (Fig. 130). E E′ the equator; let A be the position of a certain place on the globe. Take a line A C in the direction of the radius to represent the attraction of the earth, and a line A B perpendicular to the axis to represent the centrifugal force. The diagonal A D of the parallelogram on these lines will represent the resultant. This line will therefore have the direction of the plumb-line at A. When the point A is at the equator, the centrifugal force is greatest, and then is directly opposed to the resultant attraction of the earth. As A moves towards the pole A B diminishes, and makes a smaller angle with the radius. From both these causes, supposing the force A C to remain the same, the resultant A D is increased. At the pole the force A B vanishes and the resultant force is then a maximum.

From another cause the force of gravity decreases as we pass from the pole towards the equator. The polar diameter is, as we have said, shorter than the equatorial, the first being 7,926 and the second 7,899 miles, and the attracting mass is therefore nearer to a body at the pole than at the equator.

44. The centrifugal force generated in a body made *to describe* a circle is utilised in various ways, a

striking example being afforded by the governor of a steam engine (Fig. 131). X X′ is a rod which revolves

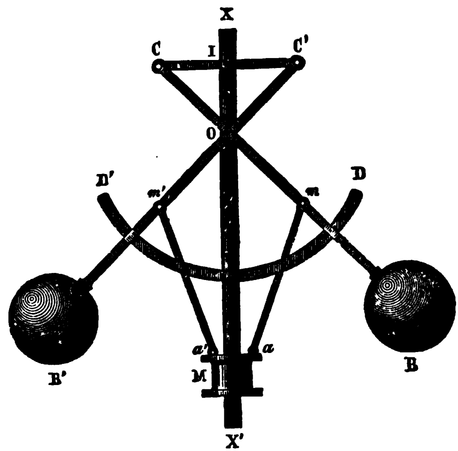


Fig. 131.

with all that is attached to it when the engine is in action; the bar C C′ and the arc D D′ are fixed to the axis, but the ring M through which the axis passes is capable of moving up and down on it.

This ring is connected with a valve which regulates the supply of steam to the cylinder. The valve is open when M occupies its lowest position, and closes as M rises. When the engine is at rest the weight of the balls B B′ brings them close to the axis, but when the machinery is in action a centrifugal force is developed, which causes them to rise until the resultant of this force and the weight acts in the direction of the rod *o m*. As the balls rise the ring M is lifted and the

supply of steam is lessened, and this is the case so long as the speed of the engine increases. If the machine moves more slowly the balls fall and the valve opens. The engine therefore regulates itself.

Exercises.

1. A railway carriage weighing 5 tons passes round a curve the radius of which is 250 yards; when the carriage is moving at the rate of 20 miles an hour, what is the outward pressure on the rails?—*Ans.* 400 lbs. nearly.

2. What ought to be the difference of level of the two rails to bring the whole pressure perpendicular to the plane of the rails, the distance between them being 57 inches?—*Ans.* $2\frac{1}{28}$ inches.

3. A weight of 3 lbs. is attached by a string, 3 feet long, to a point in a smooth horizontal table, and made to revolve round the point five times in 3 seconds; find the tension of the string.—*Ans.* $\dfrac{100\,\pi^2}{g}$.

4. Find the force towards the centre required to make a body weighing 2 lbs. move uniformly in a circle whose radius is 5 ft., with such a velocity as to complete a revolution in 5 seconds.—*Ans.* $\dfrac{8\,\pi^2}{5\,g}$.

5. A stone of 1 lb. weight is whirled round horizontally by a string $4\frac{1}{2}$ ft. long, having one end fixed; find the time of revolution when the tension of the string is 9 lbs.—*Ans.* $\dfrac{\pi\sqrt{2}}{g}$.

6. A body weighing P lbs. is at one end of a string, and a body weighing Q lbs. at the other; the system is in motion on a smooth horizontal table, P and Q describing circles with uniform velocities; determine the position of the point in the string which does not move.—*Ans.* The C. G. of P and Q.

7. A string 2 feet long can just support a weight of 16·1 lbs. *without breaking*; one end of the string is fixed to a point on a *smooth horizontal* table; a weight of 4 lbs. is fastened to the

other end and describes a circle with uniform velocity round the fixed point as centre: determine the greatest velocity which can be given to the weight so as not to break the string.—*Ans.* 16·1 ft.

8. A governor revolves once a second, the length of the rods which are jointed at a point C in the axis being 2 ft.; find the angle which they make with the vertical.—*Ans.* 66° (sin = ·9135).

9. Let the angular velocity be increased by 1-10th of its amount find the corresponding change in the angle.—*Ans.* 5° nearly (sin = ·9455).

VI.—MOMENTUM.

45. We have already shown that if P be a pressure producing the motion of a mass M, and if f be the velocity generated in a unit of time, then by taking a suitable unit of mass the relation of these quantities is expressed by the equation $P = f M$.

The product of the mass into the velocity at any instant is termed the *momentum* at that instant.

Hence, when the acceleration is uniform, $f M$ is the momentum at the end of the first second, and also the increase of momentum for every second. The above equation states that *the momentum* generated in a unit of time is proportional to the force.

The momentum bears the same relation to the moving force that the velocity does to the acceleration; the momentum varies with the time, and the moving force is the increase of momentum for a second of time.

46. When a moving body produces the motion of

another body, the greater the momentum of the first the greater the momentum generated in the second.

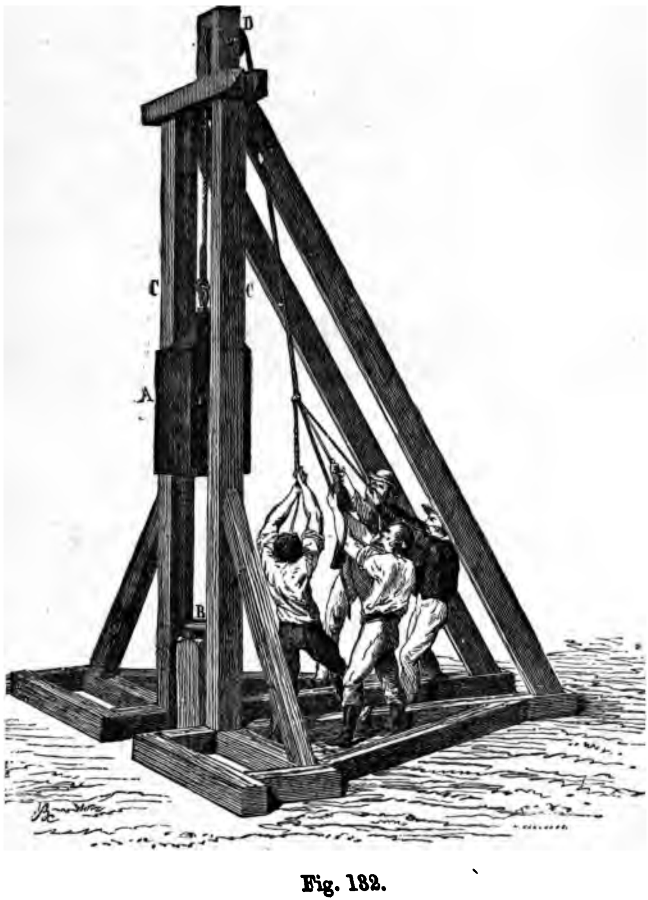


Fig. 182.

As an illustration, refer to the machine for driving piles, represented in Fig. 132. The block of iron A is raised to the top of the frame C C, and allowed to fall. It might lie on the head of the pile B for an indefinite length of time without producing any effect, but a momentum is generated in the fall which is destroyed in an exceedingly short time by the resistance of the pile. On the one hand we see that the greater the mass A, and the greater the height from which it falls, the greater is the effect, and on the other hand the greater the resistance of the pile the shorter the time during which it is overcome by the momentum of A.

Whenever the resistance to be overcome is great, the action of the force must be limited to a proportionately short time.

When a hammer is made to fall on the head of a nail the momentum of the hammer at the instant of contact is spent in overcoming the cohesion of the particles of the wood. With a given momentum the shorter the duration of the shock the greater the effect of the force in overcoming this resistance. Suppose that the board is not firmly supported so that it yields under the blow (Fig. 133), the duration of the shock is prolonged, the resistance which may be overcome is lessened, and consequently the nail drives badly. It enters much better, however, if we place behind the plank some solid body which diminishes the recoil of the board.

47. We have considered hitherto forces which act for an appreciable time, but we now see that there may be forces which act for a very brief period, and yet *in this short time* produce or destroy a great momentum.

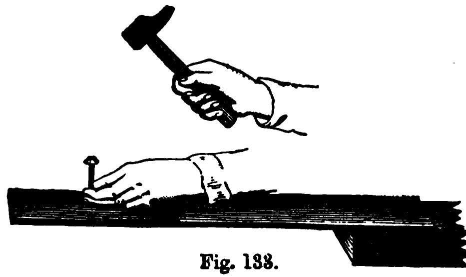


Fig. 133.

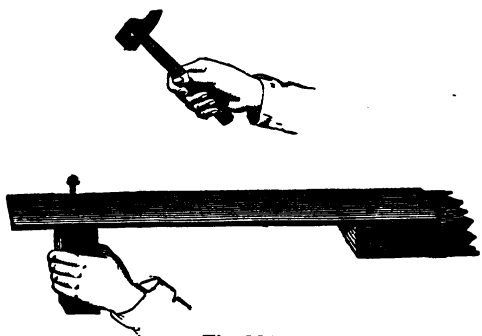


Fig. 134.

These are termed *impulsive* forces. *An impulsive force is one which produces a finite change of motion in an indefinitely short time.*

It is evident that such forces cannot be measured by the momentum generated in a unit of time; they are estimated by the whole momentum generated. Suppose, for example, that a blow given to a cricket ball of weight w drives it from the bat with a velocity v, then *the momentum* $v \times \frac{w}{g}$ is the measure of the force.

It must be remarked that the momentum of the body depends on the *mass* and not on the *weight*; a given mass of lead moving with a given velocity would strike the same blow in England as in India, although the acceleration due to gravity, and therefore the weight would not be the same in the two places.

The momentum of a moving mass at any instant is sometimes called the *accumulated force*. We have seen that a small force acting for some time may generate a momentum which if expended in a very short interval will overcome a large resistance. This principle is employed frequently in machinery. The instrument represented in Fig. 135, for stamping coins and medals is an example. A rapid motion is imparted to the weights A A′, which drives the die B on to the coin, when the accumulated momentum is destroyed in a brief interval by the resistance encountered.

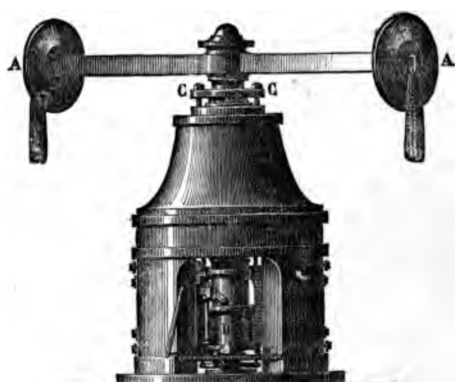


Fig. 135.

48. The first and second Laws of Motion explain the action of external forces on a body. By their means we are able to investigate the motion of a particle subjected to given forces. We have yet to examine the cases of motion which arise from the mutual action between or among two or more bodies. We shall be assisted in understanding the nature of this action by the following illustrations :—

(i.) When the block of the pile-driving engine (Fig. 132), after falling from A strikes B, it exerts a force on the pile which causes it for a very short interval of time to overcome the resistance offered by the soil. At the same time the block itself is acted on by a force upwards which brings it to rest, and these two forces are equal.

(ii.) If two balls moving in the same straight line are made to strike one another, the force of the first on the second is exactly equal to that of the second on the first, the force being measured by the momentum.

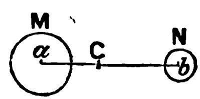


Fig. 135 *b*.

Let two balls M and N (Fig. 135 *b*) be approaching along the straight line *a b* with equal momentum, then it will be found that if they are not capable of rebounding, they will be reduced to rest after impact, the action on one being exactly equal and opposite to that on the other. Suppose, for example, that the balls weigh 6 lbs. and 8 lbs. respectively, and that their velocities are *4 ft. and 3 ft.* per second, so that the momentum of

each is $24 \div g$, then they will both be brought to rest by the impact.

It is worthy of remark that in this case the C. G. of the two balls will not change its position during the motion; for let c be the point at which the balls meet, then $a\,c$ and $b\,c$ will be described in the same time, and therefore these lines must be proportional to the velocities of the balls. But since the momenta are equal, the velocities must be inversely proportional to the masses, and consequently c divides the line $a\,b$ into parts inversely proportional to the masses. Therefore c coincides with the C.G. of the two bodies, that is to say, the C.G. is the point at which the bodies will meet, and therefore has a fixed position.

(iii.) If a cannon free to move on a smooth plane be discharged, the cannon will recoil, and its momentum will be exactly equal to that of the shot, but in the opposite direction. Let w_1 be the weight of the shot, and v_1 its velocity on leaving the gun, and let w_2 be the weight of the gun, and v_2 its velocity of recoil, then

$$w_1\,v_1 = w_2\,v_2.$$

We need not write the g in the denominators, for they cancel.

(iv.) Suppose a shell to be moving on a horizontal plane with a velocity v, and suppose it to burst in the line of its motion into two parts, whose weights are w_1 and w_2. Let the velocity of the first part after explosion be less than the original velocity by v_1, then the velocity of the second part will be greater than the original velocity by v_2, such that

$$w_1 \cdot v_1 = w_2 \cdot v_2.$$

The momentum produced in one direction by the explosion will retard the motion of w_1, and that produced in the opposite direction will be equal to the former, and will increase the velocity of w_2. The sum of the momenta of the two parts will thus be the same before the explosion as after it. This we might have concluded also from the First Law of Motion, since by that law a body cannot of itself alter its momentum.

These are illustrations of a law which is known as Newton's Third Law of Motion. *To every action there is always an equal and contrary reaction.*

The momentum produced in any direction by the mutual action of two bodies being called *action*, the momentum produced at the same time in the opposite direction is termed *reaction.*

This principle was assumed in the case of statical forces in § 7, and also in § 15, p. 8, and § 34, p. 178, where the tension in a cord at rest or in motion was supposed to be the same at every point.

Exercises on Newton's Third Law.

1. A body A of weight 10 lbs. strikes a body B at rest, and weighing 100 lbs., with a velocity of 100 ft. per second; find the velocity of B supposing A to be brought to rest by the impact.

2. Two bodies, whose weights are 6 lbs. and 10 lbs., and whose velocities are 50 ft. and 20 ft. per second respectively, approach, and after impact move on together; find the common velocity.

3. Three balls, weighing respectively 5 lbs., 7 lbs., and 8 lbs., lie in the same straight line. The first is made to impinge on the second with a velocity of 60 ft. per second without rebounding.

The first and second together impinge in the same way on the third; find the final velocity.—*Ans.* 15 ft.

4. A gun weighing 5 tons is charged with a shot weighing 28 lbs.; if the gun be free to move, with what velocity will it recoil when the ball leaves it with a velocity of 100 ft. per second?—*Ans.* ·25 ft.

5. Two wooden balls weighing 12 oz. and 16 oz. are made to impinge on one another. One of the balls is furnished with a spike to prevent the rebound. If the velocity of the smaller be 12 ft. per second, what must that of the larger be that the motion may be destroyed by the impact?

6. A shell at rest bursts into two parts, the smaller of which is one-third of the whole; what will be the ratio of the initial velocities of the parts?

7. A shell moving with a velocity of 50 ft. per second bursts in the line of its motion into two parts, which weigh respectively 30 lbs. and 62 lbs. The velocity of the larger piece is increased in the direction of motion by 30 ft. per second; what is the velocity of the smaller?—*Ans.* 12 ft. in a direction opposite to the direction of motion immediately before the explosion.

8. Two bodies subjected to their mutual attraction, and to no other forces, start from rest. If their masses are as 5 to 2, and the acceleration of the larger is 40 ft. per second, find the acceleration of the smaller.

The Fly-wheel.

49. The fly-wheel is an important contrivance for accumulating and equalising force. (Fig. 136.) It frequently happens that the work of a steam engine

Fig. 136.

is only wanted at certain intervals, and it is always the case that the force exerted by the piston is alternately stronger and weaker, and vanishes altogether at certain parts of the revolution. The momentum generated in the immense mass which forms the fly-wheel when the force is strongest, or when the resistance is weakest, carries the machinery past the points where *the force is* weakest or resistance strongest, and reduces *the variations* of speed within narrow limits.

Work.

50. The working power or efficacy of a machine or other agent is estimated by comparing it with the work required to raise a certain weight a certain height. The unit of work is the work expended in moving a weight of 1 lb. through a space of 1 foot. This unit is sometimes termed the *foot-pound*. The working power of a machine is measured by the number of units of work yielded in a given time. If, for example, the work is equivalent to the moving of a weight w through a space s, the work expended $= w \cdot s$.

Suppose the weight w to fall from a height s, there is accumulated in it $w \cdot s$ units of work. Now if v be the velocity at the end of the fall $v^2 = 2gs$ or $s = \frac{v^2}{2g}$ —hence the work accumulated is $\frac{W}{2g} v^2$. Whenever a body of weight w is moving with velocity v, there will be accumulated in it $\frac{W}{2g} v^2$ units of work; for the number of units accumulated is independent of the direction of the velocity, and if we suppose the body moving vertically upwards, it would ascend to a height $s = \frac{V^2}{2g}$, and must, therefore, have accumulated in it $w\,s$ or $\frac{WV^2}{2g}$ units of work.

Example. A train is moving at the rate of 30 ft. *per second* when the steam is cut off, if the friction be

$\frac{1}{300}$ of the weight, find how long and how far the train will move.

The number of units of work accumulated is $\frac{WV^2}{2g} = \frac{450W}{g}$

Let x be the number of feet required, then

$$x \frac{W}{300} = \frac{450\ W}{g}$$

$$\text{and } x = \frac{135000}{g} = 4218 \text{ ft.}$$

The retarding force due to friction $= \frac{g}{300}$; hence, since

$$v = f\,t,\ t = \frac{30 \times 300}{g} = 281\tfrac{1}{4} \text{ secs.}$$

Exercises.

1. A shot weighing 30 lbs. is fired from a gun weighing 3 tons, and leaves the gun with a velocity of 1,500 ft. per second; find the velocity of the gun's recoil.—*Ans.* 6·7 ft.

2. If a ball weighing 4 lbs. be thrown on a horizontal plane with a velocity of 100 ft. per second, and the friction between the ball and plane be $\frac{1}{3}$rd the weight; find the distance to which the ball will go.—*Ans.* 46·6 ft. nearly.

3. If a man were laid on a perfectly smooth table, how might he get off?

4. A weight W of 12 lbs. on a rough table is attached to a thread which passes over the edge of the table, and sustains a weight of 3 lbs.; when the latter has descended through 5 ft. the thread breaks, and W moves through 4 ft. more and comes to rest; what is the coefficient of friction?—*Ans.* $\frac{1}{6}$.

Energy.

51. Energy is a term used to denote the power of a machine or moving body to do work against some force. Energy is of two kinds, *actual* (also called *kinetic*) and potential. The actual energy of a moving body at any instant is the amount of work it is capable of doing at that instant; hence, if v be the velocity of the body, its *actual* energy is measured by the product $\frac{W V^2}{2 g}$ or $\frac{M V^2}{2}$. The actual energy of a moving body varies therefore as the square of the velocity.

When a body is thrown vertically upwards, as it rises its actual energy diminishes, and when the body reaches the highest point of its course its actual energy is spent. The body is not, however, in the same condition as at starting. If free to fall to its first position, it will acquire an actual energy exactly equal to that which has been expended in raising it. Thus the energy that was given to it has not been lost, but has been converted into an *advantage of position.* This advantage has been aptly termed *Potential Energy* (possible energy).

For instance, if a body weighing 1 lb. be projected with a velocity which would carry it vertically to a height of 100 feet, when it starts there will be 100 units of work in it; when it has passed through 60 feet there will be only 40 units of work accumulated in it. But the body being 60 feet higher than before, will have gained an advantage of position, represented by

P

60 units; thus 60 units of actual energy have been changed to potential energy, and at any instant of its flight its actual energy + its potential energy will be equal to the whole energy with which it started.

A force may be expended either in accumulating energy actual or potential, or in overcoming resistances to the motion of the body on which it acts. For instance, a horizontal force drawing a body along a smooth horizontal plane is spent in accumulating actual energy; a force pulling a body vertically upwards with accelerated velocity produces at the same time actual and potential energy; a force moving a body with uniform speed on a rough horizontal surface is spent in overcoming friction; and a force drawing a body up a rough inclined plane with accelerated velocity is expended in producing all the three effects together.

The following is the law connecting the energy acquired by a body in a given time and the forces acting on the body.

If a system of forces be applied to a machine or to a system of bodies, the work done by the forces will be exactly equal to the energy actual and potential gained by the machine supposing that there are no resistances, but if friction or other resistances have to be overcome then the work done by the forces = the actual energy + the potential energy gained + the work done in overcoming friction.

This important law was given by Newton in his comments on his Third Law of Motion.

Relation between Force and Work.

In § 45 it has been shown that the accumulated force of a moving body is measured by the product M V, that is to say, by its momentum; and in § 50 it was shown that the accumulated work is $\frac{W V^2}{2g}$ or $\frac{1}{2}$ M V^2. In the present chapter we have to consider when the effect of a moving body is to be measured by its momentum and when by its energy. The nature of the question will be better understood from the following historical sketch.

Huyghens, Wallis, Wren, and Newton, by the consideration of the collision of bodies proved that the force of a moving body is measured by the product of the mass and the velocity. The first three writers produced papers on the subject in answer to a recommendation of the Royal Society in 1668, and Newton published the results of his investigations in 1687, in which year his Principia appeared. In the following year Liebnitz published in the Leipsic Journal "the demonstration of a great error, commenced by Descartes and others in estimating the force of moving bodies." In this article it was stated that the force of a moving body did not vary as the velocity, but as the square of the velocity. In the extensive discussion which followed the publication of this paper, English mathematicians maintained that the force of a body of mass m and velocity v is proportional to $m v$, since, when two bodies meet having these products equal their motion is destroyed. The mathematicians of

Germany, Italy, and Holland, and some of those of France, held that the force of a moving body is measured by $m v^2$.* Both sides, however, always resolved the same problems in the same way, and arrived at the same results. In 1743 D'Alembert showed that both views were true, but that the term *force* was used with different significations, the one referring to time and the other to space. When a body is moving upwards against the force of the earth's attraction, the *time* during which it will rise is proportional to the velocity, but the *space* through which it will rise is proportional to the square of the velocity. (See page 160.) If a force be estimated by the time during which it will overcome a given resistance, it is proportional to v, but if by the space through which the resistance will be overcome then it is proportional to v^2. To avoid confusion it will be convenient to restrict the term *force* to the first of these meanings, using the term energy for the second, so that the *force* of a moving body will be measured by the *time* during which it will overcome a unit of resistance, and the *energy* of a moving body will be measured by the *space* through which it will overcome a unit of resistance.

The force of a moving body $= \mathrm{M\,V}$ or $\frac{\mathrm{W}}{g}\mathrm{V}$

The energy ,, ,, $= \frac{\mathrm{M\,V}^2}{2}$ or $\frac{\mathrm{W\,V}^2}{2g}$

A question *how long* is therefore one of momentum; a question *how far* is one of energy.

---

* *The unit of* mass being half that used in the preceding pages.

Example 1.—A body of mass M moves with a velocity of V feet per second, and is retarded by a constant pressure P; how long will it move?

P acting on a mass M produces an acceleration $= \frac{P}{M}$;

hence from the equation $v = V - ft$ when $v = o$,

$$\text{we have } o = V - \frac{Pt}{M}$$

$$\therefore \quad t = \frac{MV}{P}$$

$$\text{or } t = \frac{momentum}{pressure}$$

Example 2.—A body of mass M moves with a velocity of V feet per second, and is retarded by a constant pressure P; how far will it move?

The equation for the space is

$$v^2 = V^2 - 2fs$$

$$\text{where } f = \frac{P}{M}$$

hence, when $v = o$, we have

$$o = V^2 - \frac{2Ps}{M}$$

$$\therefore \quad s = \frac{MV^2}{2P}$$

$$\text{or } s = \frac{energy}{pressure}$$

Example 3.—A body moves on a rough horizontal surface, with velocity of 50 ft. per second, the coefficient

of friction being $\frac{1}{8}$; how long and how far will it move?

$$t = \frac{momentum}{resistance} = \frac{W \cdot V}{g} \div \frac{W}{8} = \frac{50 \times 8}{32} = 12\tfrac{1}{2} \text{ secs.}$$

$$S = \frac{energy}{resistance} = \frac{W\,V^2}{2\,g} \div \frac{W}{8} = \frac{2500 \times 8}{64} = 312 \cdot 5 \text{ ft.}$$

Example 4.—A train moves up an incline rising 1 in 240 with a speed of 30 miles an hour; supposing the resistances to be equivalent to one-thirtieth of the weight, find the space over which it will move after the steam is shut off.

The accumulated energy $= \dfrac{W\,V^2}{2\,g}$ where $V = 44$ ft.

Let $x =$ the space over which the train will move, then the work expended in overcoming the resistances $= \dfrac{W \cdot x}{30}$,

and the work expended in lifting the train through a height $= \dfrac{x}{240}$ is $\dfrac{W \cdot x}{240}$

$$\therefore \quad \frac{W \cdot 44^2}{64} = \frac{W \cdot x}{30} + \frac{W \cdot x}{240}$$

$$\therefore \quad x = 806\tfrac{2}{3} \text{ feet.}$$

Example 5.—A body of weight w falls down a rough plane inclined at an angle a to the horizon; if the coefficient of friction be μ, find the space passed over in t seconds.

The pressure on the plane $= W \cdot \dfrac{base}{length}$, or $W \cos a$

$\therefore$ *friction* $= \mu\, w \cos a$; the pressure exerted down

the plane by the body $= w \sin \alpha$, and the whole force down the plane $= w \sin \alpha - \mu w \cos \alpha$. This pressure acting on a weight w gives an acceleration $= g (\sin \alpha - \mu \cos \alpha)$.

Hence the equations

$$v = ft \text{ and } s = \tfrac{1}{2} f t^2$$

become $v = g (\sin \alpha - \mu \cos \alpha) t$

$$s = \tfrac{1}{2} g (\sin \alpha - \mu \cos \alpha) t^2.$$

Example 6.—In the last example if after t seconds the body commences to run up a corresponding incline, find how far it will rise.

The velocity after t seconds $= g (\sin \alpha - \mu \cos \alpha) t$,

hence the accumulated energy $= \dfrac{W \cdot g^2 (\sin \alpha - \mu \cos \alpha)^2 t^2}{2g}$.

Work done over friction $= \mu w \cdot \cos \alpha \cdot x$.

Work done in lifting the weight $= w \cdot x \sin \alpha$.

$$\therefore \quad \tfrac{1}{2} w g (\sin \alpha - \mu \cos \alpha)^2 t^2 = \mu w \cos \alpha \cdot x + w x \sin \alpha.$$

$$\therefore \quad x = \frac{g t^2 (\sin \alpha - \mu \cos \alpha)^2}{2 (\sin \alpha + \mu \cos \alpha)}$$

EXERCISES.

1. How far would a weight of 50 lbs. move against a constant resistance of 3 lbs. supposing it to start with a velocity of 120 ft. per second?

2. How long will a weight of 100 lbs. move against a resistance of 10 lbs. supposing it to start with a velocity of 20 ft. per second?

3. A railway carriage is running up an incline which rises one in sixty at a speed of 20 miles an hour; supposing the friction to be one-eightieth of the weight, find how far it will ascend the incline, and what will be its speed when it descends again to

the starting point.—*Ans.* Height $460\frac{20}{21}$ ft.; speed $\frac{88}{21}\cdot\sqrt{7}$ ft. per sec. or $20 \div \sqrt{7}$ ($= 7\cdot56$) miles per hour.

4. A weight of 400 lbs. moving on a smooth horizontal surface has a string attached to it which passes over a pulley and hangs vertically. If a weight of 10 lbs. is fastened to the cord when the larger weight has a velocity of 12 feet per second, how far will this weight move before it is brought to rest?—*Ans.* 90 feet.

5. In the above example what weight must be hung from the cord that the weight of 400 lbs. may be brought to rest in a space of 6 feet?—*Ans.* 150 lbs.

6. A certain engine if not attached to a train could get up a speed of 60 miles an hour in 5 secs.; how long would it take to get up a speed of 40 miles an hour when attached to a train of twice its own weight? The coefficient of friction is $\frac{1}{4}$, the acceleration of gravity 32 feet per second, and all resistances except friction are to be neglected.—*Ans.* 10 seconds.

7. Two bodies weighing one and two pounds respectively, are connected by an inelastic string 4 feet in length which passes over a smooth pulley; the two bodies are lifted up to the pulley, and let fall simultaneously; find the time that will elapse before the one pound weight returns to the pulley, the acceleration of gravity being 32 feet per second.—*Ans.* $4 \div \sqrt{g} = \cdot707$ secs.

8. A force capable of doing 33,000 foot-pounds of work in one minute is termed a *horse-power*. Find the horse-power of an engine which will maintain a speed of 20 miles an hour with a train weighing 80 tons, the friction being 10 lbs. per ton.—*Ans.* $42\frac{2}{3}$.

VII.—IMPACT.

Direct Impact.

Suppose a body A to impinge directly on a body B, it is a matter of observation that after impact they will be separated, A moving slower than before and B quicker; A will have lost momentum and B will have gained momentum. Now momenta lost and gained are what are termed in Newton's law (§ 35) action and reaction, and these he ascertained by numerous experiments to be equal. *Hence the momentum lost during impact by one body is equal to that gained by the other.*

The nature of the action during impact may be thus described. When A overtakes B, so long as A moves faster than B, the surfaces will be compressed, and the compression will cease when the velocities are rendered equal; if the action stops then, the bodies are said to be inelastic. Let v_1 and v_2 be the velocities of A and B before impact, and V their common velocity after impact.

The momentum lost by $A = Av_1 - AV$,

the momentum gained by $B = BV - Bv_2$;

therefore $Av_1 - AV = BV - Bv_2$;

$$\text{and } V = \frac{Av_1 + Bv_2}{A + B}$$

If the velocities be not in the same direction one must be considered negative.

Generally, however, another force comes into play *when the velocities* are equal, and the bodies begin at

that instant to recover their figure and to exert one upon the other a pressure which lasts until impact ceases. Thus A not only loses momentum during compression, but also during expansion. Let M_1 be the momentum lost by A and gained by B during compression, and M_2 that lost by A and gained by B during expansion, it is found by experiment that the ratio $\frac{M_2}{M_1}$ is constant for the same materials. This ratio, usually denoted by e, is termed the modulus of elasticity.

A body is *inelastic* when $e = 0$;
perfectly elastic when $e = 1$;
imperfectly elastic when e is between 0 and 1.

Suppose now A and B to be the masses of the two elastic bodies,

Let v_1 v_2 be the velocities before impact
V_1 V_2 ,, ,, after ,,

We have already seen that the velocity at the end of compression is

$$\frac{Av_1 + Bv_2}{A + B}.$$

and therefore the momentum lost by A up to that instant

$$= Av_1 - A \cdot \frac{Av_1 + Bv_2}{A + B} = \frac{AB(v_1 - v_2)}{A + B}$$

Therefore the momentum lost during expansion

$$= e \cdot \frac{AB\,(v_1 - v_2)}{A + B}$$

The whole momentum lost by A and gained by B is

$$\frac{(1 + e) \cdot A \cdot B \cdot (v_1 - v_2)}{A + B}$$

But the momentum lost by A is

$$Av_1 - AV_1$$

and that gained by B is

$$BV_2 - Bv_2.$$

Hence $\quad Av_1 - AV_1 = \dfrac{(1+e)\cdot A\cdot B\cdot(v_1 - v_2)}{A+B}$

and $\quad BV_2 - Bv_2 = \dfrac{(1+e)\cdot A\cdot B\cdot(v_1 - v_2)}{A+B}$

or $\quad V_1 = v_1 - \dfrac{B\cdot(1+e)\cdot(v_1 - v_2)}{A+B}$

$$V_2 = v_2 + \frac{A\cdot(1+e)\cdot(v_1 - v_2)}{A+B}$$

Oblique Impact.

52. When the direction of motion is not in the line of centres, the impact is said to be oblique. In such cases we may resolve v_1 and v_2 into velocities v_1' v_1'', and v_2' v_2'', respectively, along and perpendicular to the line of centres, then v_1'' and v_2'' will remain unchanged by the impact, and v_1' v_2' will be changed into V_1' and V_2', as if the impact had been direct and the velocities v_1' v_2''. The resultant velocities may now be found by compounding v_1'' and V_1', and v_2'' and V_2'.

Example. An imperfectly elastic body impinges obliquely on a smooth fixed plane, determine the motion after impact.

Let the velocity before be v, and the velocity after impact v'. Let the direction before impact make an angle α with the normal to the plane, and the direction after impact an angle β. Resolve these velocities in *directions* parallel and perpendicular to the plane.

The components of v are $v \sin \alpha$ and $v \cos \alpha$, and those of v' are $v' \sin \beta$ and $v' \cos \beta$. The reaction of the plane does not affect the velocity parallel with it, hence $$v \sin \alpha = v' \sin \beta. \quad \text{(i.)}$$

Let $m =$ the mass of the body, and let $\text{M} =$ the momentum lost by the body during compression in the direction of the normal. Since the plane is fixed, the momentum of the body in this direction at the end of compression is zero, and therefore

$$m\,v \cos \alpha - \text{M} = 0$$

Let $e =$ the modulus of elasticity, then the momentum generated during restitution

$$= - e\,\text{M} = - e\,m\,v \cos \alpha.$$

But this we have already seen to be

$$m \times v' \cos \beta$$

thus $$v' \cos \beta = - e\,v \cos \alpha \qquad \text{(ii.)}$$

By dividing i. by ii. we obtain

$$\cot \beta = - e \cot \alpha \qquad \text{(iii.)}$$

which determines the direction after impact.

By squaring i. and ii., and adding, we have

$$v'^2 = v^2 (\sin^2 \alpha + e^2 \cos^2 \alpha)$$

which determines the magnitude of the velocity.

Exercises.

1. An inelastic body moving with velocity v impinges on another of twice its mass at rest; find the velocity after impact.—*Ans.* $\frac{1}{3}\,v$.

2. The weights of A and B are 6 lbs. and 10 lbs., and they move in the same direction with velocities of 8 ft. and 6 ft. per second; required their velocities after impact, 1st, When the bodies are inelastic; 2nd, When they are perfectly elastic; 3rd, When the force of elasticity is $\frac{1}{3}$rd force of compression.—*Ans.* (1) 6·75 ft.; (2) 5·5 and 7·5 ft.; (3) 6·33 and 7 ft.

3. A and B weigh 12 lbs. and 7 lbs. respectively, and move in the same direction with velocities of 8 ft. and 5 ft. in a second; find the common velocity after impact; also the velocity lost by A, and that gained by B respectively.—*Ans.* $6\frac{17}{19}$, $1\frac{2}{19}$, $1\frac{17}{19}$.

4. A and B are perfectly elastic, and A, with a velocity of 20 ft. per second, strikes B at rest; find their velocities after impact, 1st, When $A = B$; 2nd, When $A = 4B$.—*Ans.* 0 and 20 ft.; 12 and 32 ft.

5. The weights of A and B are 3 lbs. and 5 lbs. and their velocities are 7 ft. and 9 ft. per second, in opposite directions; required their velocities after impact, in the same cases as in the first example.—*Ans.* (1) -3 ft.; (2) -13 and $+3$ ft.; (3) $-6{\cdot}33$ and -1 ft.

6. A, moving with a velocity of 11 ft., impinges upon B, moving in the opposite direction with a velocity of 5 ft., and by the collision A loses one-third of its momentum: what are the relative magnitudes of A and B?—*Ans.* 37, 11.

7. A, weighing 8 lbs., impinges upon B, weighing 5 lbs., and moving in A's direction with a velocity of 9 ft. in 1 second; by collision, B's velocity is trebled; what was A's velocity before impact?—*Ans.* $38\frac{1}{4}$ ft.

8. A is equal to 3B, and impinges upon B at rest; A's velocity after impact is $\frac{2}{3}$rds of its velocity before impact; required the value of e, which measures the elasticity.—*Ans.* $\frac{1}{3}$.

9. Find the elasticity of two bodies, A and B, and their proportion to each other; so that, when A impinges upon B at rest, A may remain at rest after impact, and B move on with $\frac{1}{6}$th of A's velocity before impact.—*Ans.* $e = \frac{1}{6}$; $B = 6A$.

10. At what angle must a body, whose elasticity is $\frac{1}{3}$, be incident on a plane, that the angle between the directions before and after impact may be a right angle?—*Ans.* 30°.

VIII.—CENTRAL FORCES.

53. *When a body acted on by forces tending to a fixed point is projected in any direction not passing through that point, it will move in a curve situated in one plane; and the radius from the point will describe areas proportional to the times.*

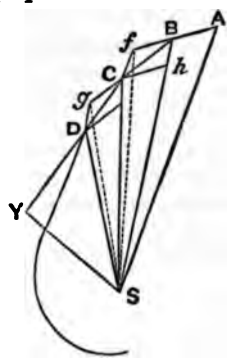


Fig. 137.

Let A be the point of projection, and let A B = the velocity of projection (Fig. 137). Then, if no force were to act on the body in one unit of time it would pass through A B, and in the next through an equal space B f in the same straight line. Suppose that the central force acting alone would carry the body through a space B h towards the centre S in a unit of time. When the body arrives at B suppose it to receive a sudden impulse towards S, which, acting alone, would carry it through B h in a unit of time. Complete the parallelogram B f C h and join S C, S f, then C will be the position of the body when it moves for a unit of time, under the action of the original velocity and the impulsive force. Since C f is parallel to S B,

$$\Delta \text{ SCB} = \Delta \text{ S}f\text{B} = \Delta \text{ SBA}.$$

In like manner if C g = the velocity at C, and C k = the space which would be passed through in a unit of time if the attracting force acted alone, the body will move to D in the next unit of time, and the

$$\Delta \text{ SDC} = \Delta \text{ S}g\text{C} = \Delta \text{ SCB} = \Delta \text{ SBA}.$$

Suppose the unit of time to be diminished (and therefore the number of units increased) indefinitely, the areas described in these units will still be equal to one another. Also the polygon A B C D will become a curve, and the attracting force will be continuous.

The radii will describe equal areas in equal times, and therefore in different times the areas will be proportional to the times.

54. *If a body move in a plain curve, so that the radius drawn to a fixed point describes areas proportional to the time, it is acted on by a central force tending to that point.*

Let B, C, D be points on the curve, and S the fixed point. Let a line A B be drawn in the direction of the velocity at B, and take A B, Bf equal to the lengths which would be described by the body in successive units of time with the velocity at B if no forces acted on it. Then A B = Bf and $\triangle$ A B S = $\triangle$ fB S.

Now imagine the body to be deflected from the straight line Bf by an impulsive force at B, which would carry it, in the given unit of time, to a point C, such that $\triangle$ C B S = $\triangle$ A B S. By the parallelogram of velocities, fC must be parallel to the direction of this impulse. But since the triangles S C B, SfB on the same base are equal, therefore fC is parallel to B S; hence the impulse must tend towards S. Similarly, if the body be made to describe the polygonal path B C D by impulses at C, D, &c., all the impulses must tend towards S.

If the number of sides be increased indefinitely, and the length of each diminished, the polygon will coincide with the curve, and the series of impulses become a continuous force which tends always to the centre.

55. *The velocity at any point* D *is inversely as the perpendicular* S Y *drawn from* S *to the tangent* D Y.

Let V = the velocity at D, t = the time of describing C D, A = the area described in a unit of time, p = the perpendicular S Y.

$$\text{Then } CD = vt, \quad \triangle\ SDC = \tfrac{1}{2}\, vt \times p; \text{ and also,}$$

$$\triangle\ SDC = At.$$

$$\therefore\ At = \tfrac{1}{2}\, vtp,$$

$$\text{and } v = \frac{2A}{p}.$$

In this equation A is constant, and consequently the velocity varies inversely as the perpendicular on the tangent.

56. That the path may be a circle, of which S is the centre, the velocity must be constant; for in this case p = the radius, and is constant.

57. From astronomical observations, it appears that all the primary planets describe ellipses, having the sun in their common focus, and that the radii describe areas proportional to the times (Kepler's Second Law). It follows, therefore, that *the primary planets are constantly acted on by forces tending towards the sun.*

58. *If a body describe an ellipse under the action of a force tending to the focus, the intensity of the force varies inversely as the square of the distance.*

Let A B C D be the path, and let D be the position of the body at a given instant (Fig. 137).

Suppose a circle described through the point D, and through two other points near to it. When these points are indefinitely near to D, the circle is called in

geometry the *circle of curvature*, and for an indefinitely small arc coincides with the curve; hence the curve and the circle have the same tangent and normal at D.*

Let ρ = the radius of curvature, let v = the velocity, and f the normal acceleration at D, then $f = \frac{V^2}{\rho}$ (§ 17).

Let F be the acceleration along D S, and let ϕ be the angle between D S and the normal at D. Resolve F along the tangent and normal, and equate the latter component to f

$$\therefore F \cos \phi = f = \frac{V^2}{\rho}$$

Now let $SY = p$, and $SD = r$,

then $$\frac{SY}{SD} = \frac{p}{r} = \cos \phi$$

also $$V = \frac{2A}{p}$$

---

* We shall have occasion to use the following geometrical facts, which the student will find demonstrated in any work on Conic Sections. They must be remembered here as properties of the curves described.

In an ellipse—

1. The perpendicular on the tangent from the focus $= b\sqrt{\frac{r}{r'}}$

2. The perpendicular on the tangent from the centre $= \frac{ab}{\sqrt{rr'}}$

3. The radius of curvature $= \frac{(rr')^{\frac{3}{2}}}{ab}$

These forms are precisely the same for a hyperbola.

In a parabola, of which $4a$ is the latus rectum, $p = \sqrt{ar}$

$$\rho = 2a\left(\frac{r}{a}\right)^{\frac{3}{2}}$$

Substituting for v and cos ϕ in the above equation, we have

$$F = \frac{4\,A^2 r}{p^3 \rho}$$

This equation is true, whatever may be the character of the curve described.

Let the curve be an ellipse, of which a and b are the major and minor semi-diameters, and let S D $= r$ and the distance of D from the other focus $= r'$.

Then it is proved in analytical geometry that

$$p = b\sqrt{\frac{r}{r'}} \text{ and } \rho = \frac{(r\,r')^{\frac{3}{2}}}{a\,b}$$

By substituting these values of p and ρ, in the general equation, we obtain

$$F = \frac{4\,a\,A^2}{b^2\,r^2}$$

It is usual to write μ for the constant part of this term; thus—let $\mu = \dfrac{4\,a\,A^2}{b^2}$ then

$$F = \frac{\mu}{r^2}$$

$$\text{or F varies as } \frac{1}{r^2}$$

59. From this proposition it follows that the attracting forces which make the planets describe ellipses having the sun in their common focus (Kepler's First Law) are inversely as the squares of the distances of the planets from the sun.

60. *A body describes an ellipse under the action of a central force in a focus; to find the time of one revolution.*

Let A = the area described by the radius in a unit

of time, and let T = the time of one revolution, then

$$T = \frac{\text{area of ellipse}}{A}$$

But the area of an ellipse $= \pi a b$, and since $\mu = \frac{4 a A^2}{b^2}$ ∴ $A = \frac{b}{2}\sqrt{\frac{\mu}{a}}$; hence

$$T = \pi a b \div \frac{b}{2}\sqrt{\frac{\mu}{a}} = \frac{2 \pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

61. *Deductions from Kepler's Third Law.*

If we consider the motion of two planets about the sun, the periodic times of which are respectively T and T′, we have

$$T^2 = \frac{4 \pi^2 a^3}{\mu} \text{ and } T'^2 = \frac{4 \pi^2 a'^3}{\mu'}$$

therefore $T^2 : T'^2 :: \frac{a^3}{\mu} : \frac{a'^3}{\mu'}$

Now, it is a matter of observation, embodied in what is known as Kepler's Third Law, that the squares of the periodic times are proportional to the cubes of the semi-major axes of the orbits, or that $T^2 : T'^2 :: a^3 : a'^3$. On comparing this proportion with the above, we see that $\mu = \mu'$. Hence the constant denoted by μ is the same for all the planets, and the attracting force of the sun does not depend on the mass of the planet attracted, but simply on its distance from the sun.

62. *If the path of a body be an ellipse, and the centre of force be the centre, then* F *varies directly as the distance.*

The perpendicular from the centre on the tangent at any point in the ellipse, distant $r\ r'$ from the foci is

proved in analytical geometry to be $\frac{a\,b}{\sqrt{r\,r'}}$
When substituted in the general equation

$$F = \frac{4\,A^2 r}{p^3 \rho} \text{ this gives}$$

$$F = \frac{4\,A^2 r}{a^2 b^2}$$

in which r is the only variable.

63. We may show in like manner that if the path be a parabola and force in the focus $F \propto \frac{1}{r^2}$ by substituting in the general equation $\rho = 2\,a \left(\frac{r}{a}\right)^{\frac{3}{2}}$ and $p = \sqrt{c\,r}$.

The values of p and ρ in the case of a hyperbola take the same form as in the ellipse; hence in this case also $F \propto \frac{1}{r^2}$

LONDON UNIVERSITY EXAMINATION PAPERS.

First B.Sc. 1870.

1. The extremities of the horizontal diameter of a circular disc, weighing 6 ounces, are nailed against a wall, and to a point in the edge of the disc at 1-12th of the whole circumference; from one of the nails, a weight of 4 ounces is attached. Find the pressure upon each nail.—*Ans.* $5 + \sqrt{3}$, and $5 - \sqrt{3}$.

2. A trapezium, having two parallel sides, which are 4 and 12 feet long, and the other sides each equal to 5 feet, is placed with its plane vertical, and with its shortest side on an inclined plane. Find the relation between the height and base of the plane when the trapezium is on the point of falling over.—*Ans.* 8 : 7.

3. State the Third Law of Motion. Show from your statement of it, how to find the tension of the string and the acceleration, when one ball is drawn up an inclined plane by another which hangs by a string passing over a fixed pulley at the top of the plane.—*Ans.* Let W and P be the weights, h and l the height and length of the plane, f the acceleration, and T the tension.

$$\text{From the motion of W, } f = g\,\frac{T}{W} - g\,\frac{h}{l}.$$

$$\text{From the motion of P, } f = g - g\,\frac{T}{P}$$

By Newton's Third Law (§ 48), the values of T in these equations are equal.

$$\therefore \quad T = \frac{P\,W\,(h + l)}{l\,(P + W)} \quad ; \; f = \frac{g\,(P\,l - W\,h)}{l\,(P + W)}.$$

4. Equal spherical inelastic bodies are placed at short equal intervals, in a smooth horizontal groove. The first is projected from the end along the groove with a velocity of 20 feet per second. Find the velocities after successive impacts.—Ans. 10, 5, 2·5, &c.

1864 to 1869.

First B.Sc.

1. Explain how a force may be represented by a straight line.

Enunciate the Parallelogram of Forces. Assuming its truth for the direction, prove the proposition for the magnitude of the resultant. (§ 23.)

If any three forces be given, can they always be applied at a point so as to balance each other?

2. Define the Centre of Gravity of a system of bodies.

Find the centre of gravity of a triangle. If a heavy point, whose weight is equal to that of the triangle, be placed at one angular point of the triangle, find the centre of gravity of the system. (§ 57.)

3. Enunciate the Third Law of Motion, and explain how far its truth is founded on experiment.

Define Velocity and Moving Force.

4. If a particle moving in a straight line with uniform acceleration f start with an initial velocity V, prove that the space passed over in any time t is $s = Vt + \frac{1}{2}ft^2$.

If f be negative, determine how far the particle will move before it comes to rest. (p. 147.)

5. Show that all the elements of forces may be represented by straight lines; and enunciate the Parallelogram of Forces. (§ 14.)

Two forces whose magnitudes are as 3 to 4 act on a point in directions at right angles, and produce a resultant of 2 lbs.; find the forces.—*Ans.* 1·2 and 1·6 lbs.

6. In a system of pulleys in which the same string passes round all the pulleys, and the parts of it between the pulleys are parallel, show that there is equilibrium when the power is to the weight as unity is to the number of strings at the lower block. (§ 87.)

A man supports a weight equal to half his own weight by such a system of pulleys, of which the upper block is attached to the ceiling. If there be seven strings at the lower block, find his *pressure on* the floor on which he stands.—*Ans.* $\frac{13}{14}$ of his weight.

7. Define the centre of gravity of a body; and show that a heavy body has one centre of gravity, and only one. (§§ 55, 42.)

A square is divided into four equal squares by straight lines drawn parallel to its sides. If one of the four squares be removed, find the centre of gravity of the remaining figure. (§ 58.)

8. When a particle is projected with a given velocity a in a direction contrary to that in which a force f acts, find its velocity v after describing a given space s. (p. 156.)

9. Show that if a particle be projected in any direction not vertical, and acted on by gravity, it will describe a parabola. (p. 167.)

10. Enunciate the Parallelogram of Forces. Assuming that it is true for forces P and Q acting at a certain angle, show that it is true for forces P and 2 Q acting at the same angle. (§ 22, 3rd.)

11. In a system of pulleys in which each pulley hangs by a separate string, and the strings are parallel, show that there is equilibrium when P is to W as 1 is to 2^n where n is the number of movable pulleys, the weight of each pulley being neglected. (Fig. 78.)

12. If there are three pulleys in such a system, and the weight of each is w, show that $P - w$ is to $W - w$ as 1 is to 8.

13. Show how to find the centre of gravity of a body when it is made up of two parts and the centre of gravity of each part is known. A, B, C are three heavy particles in known positions. Having given the centre of gravity of the three particles, find the centre of gravity of any two of them. (Fig. 43.)

14. Determine the motion of a particle projected down a smooth inclined plane.

Show that the times of descent down all chords of a circle, measured from the highest point, are the same. (p. 183.)

15. Determine the straight line down which a particle must move in order to pass in the least time from a given point to a given straight line. (p. 184.)

16. Derive from the parallelogram of forces the conditions of equilibrium of three parallel forces applied to astraight rod. *(§ 39.)*

17. Find the ratio between the power and the weight in the case of the screw, and state some of the most useful applications of this power. (§ 91.)

18. The two boxes in Attwood's machine are so adjusted that one contains $3\frac{1}{2}$ oz. and the other $2\frac{1}{2}$ oz. How long will the heavier take to fall 1 ft.? Also what will be the tension of the string during the motion?—*Ans.* $\sqrt{\frac{12}{g}}$ or $\frac{1}{4}\sqrt{6}$ secs.; tension $= 2\frac{11}{12}$ oz.

19. A man in the act of being weighed in a balance of the ordinary kind, pushes with a walking-stick the beam of the balance at a point A between the point of suspension, S, of the scale in which he is, and the fulcrum F. What effect, if any, will be produced on his apparent weight? If the scale in which the man is be kept from moving laterally by a horizontal string attached to a fixed point, what will be the effect?

Solution.—If C be the scale and A C the rod, there is a pressure P at A along C A, and an equal pressure at C along A C. Resolve into horizontal and vertical components H V. V V is a couple having a moment V · S F — V · A F or V · A S tending to turn the beam towards C. The component H at A is counteracted by the reaction of F, but H at C tends to push out the scale. Thus both components cause C to descend. The horizontal string would counteract the lower H, but would not affect the action of the couple V V.

20. When there is equilibrium in that system of pulleys in which one end of the string passing round each pulley is attached to a fixed support, and the system is displaced, show that the power is to the weight as the space through which the weight is lifted is to the space through which the power is moved, the weights of the pulleys being neglected. (p. 117.)

21. A parallelogram is divided along a diagonal; and one half remaining fixed, the other half is lifted, reversed, and applied to the former half along the same diagonal. Find the distance between the C. G. of the quadrilateral figure thus formed and of the original parallelogram.—*Ans.* 1-3rd of the distance between the centre and the perpendicular from the opposite angle.

22. Define the mass of a body; and describe an experiment

which shows the masses of bodies are proportional to their weights, giving fully the steps of the argument. (p. 175.)

23. An equilateral triangle is placed with one side vertical; and two equal masses, connected by a slack inelastic string which passes without friction over the upper corner, are allowed to fall from the upper angle, the first down the slant side, the second down the vertical side. What start in time must be given to the first mass that when the string is pulled tight the masses may destroy each other's velocity?—*Ans.* Since the masses are equal the velocities must be equal in order that the momenta may be destroyed when the string becomes tight; hence the two masses must then be in the same horizontal line. Let $h =$ the vertical, and $l =$ the inclined space passed through, then the acceleration down $h = g$, and that down $l = g\frac{h}{l}$; hence the time down $h = \sqrt{\frac{2h}{g}}$, and the time down $l = \sqrt{\frac{2l^2}{gh}}$. Let L $=$ the length of the string $= h + l$; since the angle between h and l is 60°, therefore $l = 2h$, hence $h = \frac{1}{3}$ L. The difference of the times is therefore $\frac{\sqrt{2L}}{\sqrt{3g}}$.

24. A uniform equilateral triangle A B C has a sphere of the same weight as the triangle attached to it, so that the centre of the sphere is at the angular point C of the triangle. If the whole be suspended from the middle point of A C, find the inclination of the sides of the triangle to the string.—*Ans.* A C and A B at 30°, and B C at 90°.

25. A horse walking by the side of a canal is to draw a boat along the canal by means of a horizontal rope attached to the boat near the bow. Point out the position in which the rudder must be placed in order that the boat may move parallel to the bank: and show in a diagram *all* the forces acting upon the boat when in motion.—*Ans.* Resolve the tension of the cord into two parts, one in the direction of the axis of the boat and the other perpendicular to it. The latter tends to turn the bow towards the

land, and this tendency will be counteracted by the pressure on the rudder if the rudder be turned from the land.

26. To each end of a uniform straight rod, A B, 100 inches long, and weighing 12 lbs., is fastened one end of a flexible string A C B, 140 inches long, to which a weight of 9 lbs. is attached at a point C, 60 inches from one end. In what position will the rod remain in equilibrium about a pivot through the middle? and where must the pivot be placed in order that the rod may be balanced when horizontal?—*Ans.* The ratio of the sides 3 : 4 : 5 shows that the $\llcorner$ C = 90°. Let M be the middle of A B, and N the foot of the perpendicular from C. From the proportion N A : A C :: A C : A B we obtain A N = 36; and therefore M N = 14, N C = 48. When M is the point of suspension M C is vertical, and the rod makes angle A M C, which is determined by the ratio $\frac{CN}{NM} = \frac{48}{14}$ (tan), with the vertical. When the rod is horizontal the point of suspension must be the centre of the forces, 9 lbs. at N and 12 at M, that is, at a point 6 inches from M.

27. A man sitting upon a board suspended from a single movable pulley, pulls downwards at one end of a rope which passes under the movable pulley and over a pulley fixed to a beam overhead, the other end of the rope being fixed to the same beam; what is the smallest proportion of his whole weight with which the man must pull in order to raise himself?—*Ans.* A little more than one-third.

28. With what force would he require to pull upwards if the rope, before coming to his hand, passed under a pulley fixed to the ground, as well as round the other two pulleys?—*Ans.* A little more than his own weight.

29. The last carriage of a railway train gets loose whilst the train is running at the rate of 30 miles an hour up an incline of 1 in 150. Supposing the effect of friction upon the motion of the carriage to be equivalent to a uniformly retarding force equal to $\frac{1}{300}$ of the weight of the carriage; find—(I.) the length of *time during* which the carriage will continue running up the *incline*, and (II.) the velocity with which it will be running

down after the lapse of twice this interval from the instant of its getting loose.—*Ans.* The acceleration due to friction (§ 32) $= -\frac{F}{m} = -\frac{1}{300}\frac{W}{m} = -\frac{g}{300}$; and that due to gravity (§ 37) $= -g\frac{h}{l} = -\frac{g}{150}$

Hence their sum is $-\frac{g}{100}$

The equation of motion is

$$v = V - \frac{g}{100}t,$$

but when the train stops $v = o$.

$$\therefore t = \frac{100\,V}{g}.$$

Since $V = 30$ miles an hour, or 44 ft. per sec., and $g = 32$ $\therefore$ $t = 137{\cdot}5$ secs.

In descending the acceleration $= \frac{g}{150} - \frac{g}{300} = \frac{g}{300}$ and the velocity in t secs. $=$

$$t \times \frac{g}{300} = 137{\cdot}5 \times \frac{32}{300} = 14{\cdot}\dot{6}.$$

30. How is the energy of a moving body estimated? Through what distance must a force equal to the weight of ½ lb. act upon a mass of 48·3 lbs. in order to increase its velocity from 24 ft. to 36 ft. per second?—*Ans.* 1080.

Second B.A. and Second B.Sc.

1. Prove the truth of the Parallelogram of Forces for the *direction* of the resultant. (§ 22.)

The directions of two forces acting at a point are inclined to

each other (1) at an angle of 60°, (2) at an angle of 120°; and the respective resultants are in the ratio $\sqrt{7} : \sqrt{3}$. Compare the magnitudes of the forces.—*Ans.* 2 : 1. (§ 95.)

2. Define the Centre of Gravity of a body or system, and find that of three weights placed at the angular points of a triangle, (1) when the weights are equal, and (2) when they are unequal. (§ 58, 43.)

In either case, if h, k, l be the distances of the weights from the opposite sides of the triangle, and x, y, z the respective distances from the same sides of the centre of gravity, prove that

$$\frac{x}{h} + \frac{y}{k} + \frac{z}{l} = 1.$$

3. Find the relation between the power and the weight when there is equilibrium in the system of pulleys in which each hangs by a separate string, neglecting the weights of the pulleys. (§ 86.)

If the power be drawn downwards at a given rate, at what rate will the weight ascend? (§ 94.)

4. Explain what is meant by Uniform Acceleration; and if a point move from rest with a constant acceleration f, prove that the space described in a time t is $\frac{1}{2} f t^2$. (§ 15.)

A heavy particle is placed on a smooth inclined plane and allowed to slide down; determine its motion, stating the laws of motion assumed in the investigation. (§ 37.)

5. Prove that a heavy particle projected in vacuo in a direction not vertical will describe a parabola. (§ 24.)

A ball is projected in a direction inclined at an angle of 30° to the horizon, and with the velocity which it would have acquired in falling from rest through a space of 100 yards; find the greatest height attained by the ball.—*Ans.* 75 ft. (§ 26.)

6. A body describes an orbit round a fixed centre of force; state the relation between the time and the area swept over by the radius vector. (§ 53.)

Will this relation be affected by any sudden change in the law *of force?*

What is the law of force when the orbit is an ellipse and the centre of force is at one of the foci?

7. Define the resultant of two forces; and explain why forces may be represented by straight lines. (§ 14.)

Enunciate the Parallelogram of Forces; and, assuming it for the direction, prove it for the magnitude of the resultant. (§ 23.)

What is the resultant of two equal forces inclined to each other at an angle of 120°? (Fig. 14.)

8. Find the centre of gravity (1) of the area, (2) of the perimeter of a triangle. (§§ 57, 59.)

Three equal rods are jointed together so as to form an equilateral triangle A B C. If D, E be the middle points of the rods A B, A C, prove that the distance between the centres of gravity of the portions D A E, E C B D of the rods is $\frac{3}{8}$ths of the altitude of the triangle.

9. Describe the construction and action of a screw; and find the relation between the power and the weight when there is equilibrium.

10. Explain how acceleration is measured; and state the amount of the acceleration due to the action of gravity.

11. A heavy body is let fall from rest *in vacuo*; how far does it fall in the first, second, and fifth seconds respectively?

12. A body is projected vertically upward with a velocity of 89 ft. per second; in what time will it attain a height of 149 ft.? —*Ans.* Never; the greatest height is 123 ft.

13. Define a Central Force; and explain, in general terms, how a particle acted upon by a central force describes a curvilinear orbit.

Of what nature are the orbits described by the planets about the sun? and what is the relation between their periodic times and mean distances? (§§ 53, 61.)

14. A B C D is a square; a force of 1 lb. acts along the side A B from A to B; a force of 1 lb. acts along the side A D from A to D; and a force of 2 lbs. acts along the side C B from C to B; determine the magnitude and position of the resultant of the three forces.—*Ans.* $\sqrt{2}$ at C parallel to D B.

15. Define the centre of gravity of a body, or a system of bodies.

Show how to find the position of the centre of gravity of a system of heavy particles.

Weights are placed at the corners of a triangle, each weight being proportional to the length of the opposite side; show that the centre of gravity of the weights is situated at the centre of the circle inscribed in the triangle. (§ 59.)

16. Enunciate the First and Second Laws of Motion.

17. A particle describes a circle under the influence of a force always tending to the centre of a circle; show that the velocity of the particle is constant; and find the relation between the force, the velocity, and the radius of the circle. (§ 56 and 17.)

18. On what experimental facts do you consider statics to be founded? Deduce from these the Parallelogram of Forces.

19. Three forces act on a point P within a triangle, and are proportional to the distances of P from the angular points, and act directly from these points; find the position of P when those forces are in equilibrium.—*Ans.* The C. G. of the triangle.

20. Show that if two forces acting on a lever which can turn freely about a fixed point be in equilibrium, the moments of the forces about the fixed point are equal. (Two cases: 1st, when the forces meet; 2nd, when parallel. §§ 46, 47.)

21. Two weights, W and W′, are carried on a pole which rests on the shoulders of two men, A and B, of equal height. The weight W rests at a point C, such that AC : CB :: 3 : 2; and W′ at a point D, such that AD : DB :: 5 : 2; find the proportion of the weights borne by the two men.

$$\textit{Ans.}\ \text{A},\ \frac{2}{5}\,\text{W} + \frac{2}{7}\,\text{W}';\ \text{B},\ \frac{3}{5}\,\text{W} + \frac{5}{7}\,\text{W}'.$$

22. Define the centre of gravity of a solid body. Find the centre of gravity of a uniform triangle.

A particle whose weight is equal to that of the triangle is attached to one of the angular points, and the whole system is suspended by a string from a fixed point. To what point of the triangle should the string be attached that the triangle may rest with its plane horizontal?—*Ans.* Midway between the C. G. and the particle.

23. Find the time of descent from rest down a plane of given length inclined at a given angle to the horizon.—*Ans.* $t = \sqrt{\dfrac{2l}{g \sin \alpha}}$.

24. A particle slides from rest down any diameter of a vertical circle. Show that the velocity acquired in the descent varies inversely as the time of descent. (§§ 38, 39.)

25. A particle is projected in vacuo in a given direction with a given velocity. Find the range on a horizontal plane through the point of projection. (§ 28.)

26. Two particles P and Q are projected at the same instant from the same point, one with velocity V at an angle of 60° to the horizon, and the other with velocity $\dfrac{V}{\sqrt{3}}$ at an angle of 30°. Will they ever hit each other?—*Ans.* The condition that the equations $y = x \tan \alpha - \frac{1}{2} gt^2$ and $y = x \tan \alpha' - \frac{1}{2} gt^2$ are satisfied, for the same value of x, y, and t, is that $\tan \alpha' = \tan \alpha$; but this is not the case with the given data.

27. If the earth be supposed to describe an elliptic orbit of eccentricity $\dfrac{1}{60}$ about the sun considered as a point, compare the velocities of the earth when nearest to and farthest from the sun; also compare the angular velocities of the earth at the same points as seen from the sun.—*Ans.* $\dfrac{1}{59} : \dfrac{1}{61}$

28. If three forces acting on a particle be represented in magnitude and way of action by the sides of a triangle taken in order, show that they will keep the particle in equilibrium.

29. Find the C. G. of a cone. From a given cone a cone of half the height is cut off by a plane section parallel to the base. Find the C. G of the remaining segment.—*Ans.* $\dfrac{11}{56} h$ from the base.

30. Find the conditions of equilibrium on the wheel and axle.

31. Find the time of falling from rest down a chord of a circle drawn from the highest point. Find the straight line of quickest *descent from* a given point to a given plane.

32. A particle moves under the influence of any number of forces which are invariable in magnitude and direction. Show that the particle must move either in a straight line or in a parabola. Find the time in which a projectile reaches its greatest height, and the greatest height. (§§ 26, 28.)

33. When a body moves under the action of a central force, the areas described by the radius to the centre of force are in one plane and are proportional to the times of describing them. (§ 53.)

34. If three forces, acting in one plane on a body, be in equilibrium, prove that their lines of action are either parallel or meet in a point. State also all the other conditions necessary for equilibrium.

35. A heavy equilateral triangle is placed with its plane vertical, and one side resting on a rough inclined plane; the coefficient of friction being $\sqrt{3}$; what is the greatest inclination of the plane to the horizon that the triangle may neither slide down the plane nor roll over an angular point?—*Ans.* 60°.

36. Define the C. G. of a body. Show that the C. G. of any triangle is the same as that of three equal particles placed one at each angular point. (§ 58.)

37. Define Uniform Acceleration. A particle moves with a uniform acceleration f. If V be the velocity at any instant, show that the space described in time t after that instant is $V t + \frac{1}{2} f t^2$. How is it shown that the acceleration due to gravity is uniform? (§ 15.)

38. A particle slides down a smooth inclined plane under the action of gravity. If the particle start from rest, find the time of describing a given space.

39. A B is the vertical diameter of a circle whose plane is vertical; P Q is a diameter inclined to A B at an angle θ; find θ, that the time of falling down P Q may be twice that of falling down A B. (The formulæ are $S = \frac{1}{2} g t^2$ and $S = \frac{1}{2} g \cos \theta \, t'^2$) —*Ans.* $\cos \theta = \frac{1}{4}$.

40. A particle is projected from a given point in vacuo under the action of gravity. From any point P a tangent P T is drawn to touch the path in T, and a vertical line P Q is drawn to cut

the path in Q. If V be the velocity of the particle at T, prove that $\frac{PT^2}{PQ} = 2\frac{V^2}{g}$ (p. 168).

41. If a particle describe an orbit about a centre of force, prove that the velocity at any point is inversely proportional to the perpendicular from the centre of force on the tangent at that point. (§ 55.)

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1. (α) If three forces in the same plane, but not in the same right line, acting on a point, are in equilibrium, prove that each is proportional to the sine of the angle contained between the directions of the other two. (§ 96.)

(β) If three forces acting on a point are not in the same plane, prove that they cannot be in equilibrium. (§ 34.)

2. (α) Determine the ratio between the power and weight for equilibrium on the inclined plane. (§ 88.)

(β) In a screw-press state what represents the weight, technically so-called, and the ratio between it and the power.

3. A triangular slab is supported in the air by means of three vertical strings attached to its angular points. Prove that in whatever way the weight of the slab is distributed, the tension of each string is independent of the position in which the slab is held, and that if the weight is distributed uniformly over the surface, the tension of each string will be one-third of the weight. (§ § 43, 58.)

4. (α) Why is it that we are not conscious of the motion of revolution of the earth round the sun, and that we can perform the common actions of life, such as walking, riding, or carrying the food to our mouths, without reference to the direction in which the earth may happen to be moving? (§ 19, p. 154.)

(β) Supposing that the weight of a heavy free piston can be just supported by the pressure of steam in a cylinder held in an upright position, compare its motion in the cylinder (neglecting friction) with that of a stone falling in vacuo—(1) when the cylinder *is brought* into a horizontal, and (2) into an inverted *vertical position*; the steam being supposed to be maintained in *each case* at a uniform pressure. (§ 33, p. 177.)

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